PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 3, March 1981

SPECTRA OF OPERATORS WITH FIXED IMAGINARY PARTS

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ABSTRACT. The aim of this paper is to obtain the best bound for the distance between the eigenvalues of a Hermitian matrix B and the real parts of eigenvalues of a matrix $B + iA$, where A is also Hermitian, in the terms of eigenvalues of A. A similar problem in infinite-dimensional Hilbert space is also considered.

This paper was inspired by the papers of Kahan **[4]**, **[5]** and Gohberg **[1]**. The obtained results may be regarded as the generalizations of some results of these authors. A solution of the problem of Kahan, which consists of computing the best constant in the inequality $\|Z - Z^*\| < K_n\|Z + Z^*\|$ for all $n \times n$ matrices with real spectrum, is obtained (Corollary 2).

Notations. Let H denote a complex Hilbert space with the norm $\|\cdot\|$ and the scalar product $\langle \cdot , \cdot \rangle$. $L(H)$ denotes the algebra of all bounded linear operators acting in H. For an $A \in L(H)$, $\sigma(A)$ denotes the spectrum of A. For a compact $A \in L(H)$, $s_1, s_2, \ldots$ denote the eigenvalues of $\sqrt{AA^*}$, repeated according to multiplicity and arranged in decreasing order.

Finite-dimensional case. In this section we assume that H is n-dimensional and that $A = A^*$ is an operator in H with eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_n$.

LEMMA 1. *There exists $B = B^* \in L(H)$ such that $\sigma(A + iB) \subset \mathbf{R}$ and*

$$\|B\| = \frac{1}{n}\sum_{j=1}^{n} \lambda_j \operatorname{ctg} \frac{2j-1}{2n}\pi.$$

PROOF. Let $\{e_j\}_1^n$ be an orthonormal basis of H. Define operator S by the formula

$$S = -\langle \cdot , e_1\rangle e_n + \sum_2^n \langle \cdot , e_j\rangle e_{j-1}.$$

The vectors $v_k = \sum_{j=1}^n \exp(((2k-1)/n)j\pi i)e_j$ are mutually orthogonal eigenvectors of S of norm n. When the basis $\{e_j\}$ is suitably chosen then

$$A = \frac{1}{n}\sum_1^n \lambda_j\langle \cdot , v_j\rangle v_j.$$

Received by the editors May 25, 1979.
AMS (MOS) subject classifications (1970). Primary 15A42, 15A60, 47A55; Secondary 15A18, 15A57.

We shall show that the operator

$$B = i \sum_{k,j=1}^{n} \operatorname{sign}(k-j)\langle Ae_j, e_k\rangle\langle \cdot , e_j\rangle e_k$$

satisfies the thesis of our lemma.

Since $A = \sum_{k,j=1}^{n}\langle Ae_j, e_k\rangle \langle \cdot , e_j\rangle e_k$ we see that the matrix of the operator

$$A + iB = \sum_{k=1}^{n} \langle Ae_k, e_k\rangle\langle \cdot , e_k\rangle e_k + 2\sum_{k<j} \langle Ae_j, e_k\rangle\langle \cdot , e_j\rangle e_k$$

is triangular in the basis $\{e_j\}$. Therefore the eigenvalues of $A + iB$ are $\lambda_k(A + iB) = \langle Ae_k, e_k\rangle = (1/n)\sum_1^n\lambda_j$. Hence $\sigma(A + iB) \subset \mathbf{R}$.

Note that

$$\langle Ae_j, e_k\rangle = \left\langle \frac{1}{n}\sum_{s=1}^{n} \lambda_s\langle e_j, v_s\rangle v_s, e_k\right\rangle = \frac{1}{n}\sum_{s=1}^{n}\lambda_s \exp\left(\frac{2s-1}{n}(k-j)\pi i\right).$$

Now it is easy to verify that if we define the numbers b_j and the operator T by the relations

$$b_j = -i\sum_{s=1}^{n}\lambda_s \exp\left(-\frac{2s-1}{n}j\pi i\right), \qquad T = \langle \cdot , e_1\rangle e_n + \sum_{k=1}^{n-1}\langle \cdot , e_{k+1}\rangle e_k,$$

then $B = (1/n)\sum_{j=1}^{n-1} b_j T^j$. Since the numbers $\lambda_k(T) = \exp(2k\pi i/n)$ are the eigenvalues of T, the eigenvalues of B are

$$\lambda_k(B) = \frac{1}{n}\sum_{j=1}^{n-1} b_j(\lambda_k(T))^j = \frac{1}{n}\sum_{s=1}^{n}\lambda_s \operatorname{ctg}\frac{2(s-k)-1}{2n}\pi.$$

Hence

$$\|B\| = \max_k|\lambda_k(B)| = \frac{1}{n}\sum_{s=1}^{n}\lambda_s \operatorname{ctg}\frac{2s-1}{2n}\pi.$$

LEMMA 2. *If $B = B^* \in L(H)$ and $\sigma(A + iB) \subset \mathbf{R}$ then*

$$\|B\| \leqslant \frac{1}{n}\sum_{s=1}^{n}\lambda_s \operatorname{ctg}\frac{2s-1}{2n}\pi.$$

PROOF. By the theorem on triangular matrix form there exists an orthonormal basis $\{e_j\}_1^n$ of H such that $\langle (A + iB)e_k, e_j\rangle = 0 = \langle (A - iB)e_j, e_k\rangle$ for $k < j$. This implies that $i\langle Be_j, e_k\rangle = \langle Ae_j, e_k\rangle$ and $i\langle Be_k, e_j\rangle = -\langle Ae_k, e_j\rangle$ for $k < j$. Since $\langle A + iBe_k, e_k\rangle \subset \sigma(A + iB) \subset \mathbf{R}$ and A, B are selfadjoint, $\langle Be_k, e_k\rangle = 0$. Hence $B = \sum_{j,k}\langle Be_j, e_k\rangle\langle \cdot , e_j\rangle e_k = -\sum_{j,k} \operatorname{sign}(j-k)\langle Ae_j, e_k\rangle\langle \cdot , e_j\rangle e_k$. Setting $\langle \cdot , e_j\rangle e_j = E_j$ we may write

$$B = i\sum_{j,k} \operatorname{sign}(k-j)E_kAE_j. \tag{1}$$

Since $B = B^*$ there is a unit eigenvector f of B such that $\|B\| = |\langle Bf, f\rangle|$. If we set $\langle \cdot , f\rangle f = F$ then $\operatorname{tr} BF = \langle Bf, f\rangle$. Using properties of trace and (1) we see that

$$(\pm\|B\| =) \operatorname{tr} BF = \operatorname{tr}\left(i\sum_{j,k} \operatorname{sign}(k-j)E_kAE_jF\right)$$

$$= \operatorname{tr}\left(A\sum_{j,k} i \operatorname{sign}(k-j)E_jFE_k\right) = \operatorname{tr} AG, \tag{2}$$

where we have set $i\sum_{j,k}\operatorname{sign}(k-j)E_jFE_k = G$. G is a selfadjoint operator. Let $\omega_1 \geqslant \omega_2 \geqslant \cdots \geqslant \omega_n$ be its eigenvalues. It is shown in [1] that

$$\omega_j = -\omega_{n+1-j}, \qquad j = 1, 2, \ldots, n, \tag{3}$$

and that if $\|E_jf\| \neq 0$ for all j then for $j \leqslant [n/2]$ $\sum_{k=1}^{n}\arg(\omega_j + i\|E_kf\|^2) = (2j-1)\pi/2$. This means that

$$\frac{(2j-1)\pi}{(2n)} = \frac{1}{n}\sum_{k=1}^{n} \operatorname{arc\,tg}\left(\|E_kf\|^2/\omega_j\right)$$

$$\leqslant \operatorname{arc\,tg}\left(\frac{1}{n}\sum_{k=1}^{n}\|E_kf\|^2/\omega_j\right) = \operatorname{arc\,tg}\frac{1}{n\omega_j},$$

since the function arc tg is concave in the interval $[0, \infty]$. Since tangent is an increasing function in $[0, \pi/2)$ we obtain the inequality $\operatorname{tg}((2j-1)\pi/(2n)) \leqslant 1/(n\omega_j)$, equivalent to

$$\omega_j \leqslant \frac{1}{n}\operatorname{ctg}\frac{2j-1}{2n}\pi, \qquad j = 1, 2, \ldots, n/2. \tag{4}$$

By continuity of eigenvalues (4) holds also when $E_jf = 0$ for some j.

It follows from (3) that $\operatorname{tr} G = 0$. Let $\mu > -\lambda_n$; then

$$s_j(A+\mu) = \lambda_j + \mu. \tag{5}$$

Let x_j be the normalized eigenvector of G, $Gx_j = \omega_j x_j$. Since G is selfadjoint $\{x_j\}_1^n$ is an orthonormal basis for H. Thus, using Abel transformation, we may write

$$\operatorname{tr} AG = \operatorname{tr}(A+\mu)G = \sum_j \langle (A+\mu)Gx_j, x_j\rangle = \sum_j \omega_j\langle (A+\mu)x_j, x_j\rangle$$

$$= \sum_{j=1}^{n-1}(\omega_j - \omega_{j+1})\sum_{k=1}^{j}\langle (A+\mu)x_k, x_k\rangle + \omega_n\sum_{k=1}^{n}\langle (A+\mu)x_k, x_k\rangle. \tag{6}$$

The Ky-Fan lemma [2, Lemma II.4.1] and (5) imply that

$$\left|\sum_{k=1}^{j}\langle (A+\mu)x_k, x_k\rangle\right| \leqslant \sum_{k=1}^{j} s_k(A+\mu) = \sum_{k=1}^{j}(\lambda_k+\mu).$$

Note also that if $j = n$ then in the above inequality we have in fact the equality without the modulus. Hence

$$\operatorname{tr} AG \leqslant \sum_{j=1}^{n-1}(\omega_j - \omega_{j+1})\sum_{k=1}^{j}(\lambda_k+\mu) + \omega_n\sum_{k=1}^{n}(\lambda_k+\mu)$$

$$= \sum_{j=1}^{n}\omega_j(\lambda_j+\mu) = \sum_{j=1}^{n}\lambda_j\omega_j.$$

Writing the just obtained inequality with $-A$ instead of A we obtain by (3) the inequality $-\operatorname{tr} AG \leqslant \Sigma_j\omega_j(-\lambda_{n+1-j}) = \Sigma_j\lambda_j\omega_j$. This with (2), (3) and (4) shows that

$$\|B\| = |\operatorname{tr} AG| \leqslant \sum_j \lambda_j\omega_j = \sum_{j=1}^{[n/2]} (\lambda_j - \lambda_{n+1-j})\omega_j$$

$$\leqslant \frac{1}{n}\sum_{1}^{[n/2]} (\lambda_j - \lambda_{n+1-j})\operatorname{ctg}\frac{2j-1}{2n}\pi = \frac{1}{n}\sum_{1}^{n}\lambda_j \operatorname{ctg}\frac{2j-1}{2n}\pi.$$

The lemma is proved.

THEOREM 1. *Suppose that B is a selfadjoint operator in H. Let $\{\beta_j\}_1^n$, $\{\mu_j\}_1^n$ be the eigenvalues of B, $B + iA$, respectively, arranged in such a way that $\beta_j \geqslant \beta_{j+1}$, $\operatorname{Re}\mu_j \geqslant \operatorname{Re}\mu_{j+1}$ for $j = 1, 2, \ldots, n-1$. Then*

$$|\beta_j - \operatorname{Re}\mu_j| \leqslant \frac{1}{n}\sum_{s=1}^{n}\lambda_s \operatorname{ctg}\frac{2s-1}{2n}\pi.$$

PROOF. Following Kahan [5] and identifying the operators with matrices we may assume that $B + iA$ is an upper triangular matrix and that $B + iA = D + iZ$, where D is a real diagonal matrix, Z is an upper triangular matrix with real spectrum. Hence the numbers $\operatorname{Re}\mu_j$ are eigenvalues of D. Since $B - D = i(Z - Z^*)/2$ it follows from Weyl's inequality that $|\beta_j - \operatorname{Re}\mu_j| \leqslant \|B - D\| = \|\operatorname{Im} Z\|$. Since $\operatorname{Re} Z = (Z + Z^*)/2 = A$ the thesis follows from Lemma 2.

For a subset F of the complex plane let $\operatorname{Re} F = \{\operatorname{Re}\lambda;\ \lambda \in F\}$. The following corollaries follow easily from the obtained results.

COROLLARY 1.

$$\max_{B=B^*\in L(H)} \operatorname{dist}(\sigma(B), \operatorname{Re}\sigma(B + iA)) = \frac{1}{n}\sum_{1}^{n}\lambda_s \operatorname{ctg}\frac{2s-1}{2n}\pi.$$

($\operatorname{dist}(\cdot, \cdot)$ *denotes the Hausdorff distance of sets.*)

COROLLARY 2.

$$K_n = \max\{\|Z - Z^*\|/\|Z + Z^*\|;\ Z \in L(H), \sigma(Z) \subset \mathbf{R}, Z \neq 0\}$$
$$= \frac{2}{n}\sum_{1}^{[n/2]}\operatorname{ctg}\frac{2s-1}{2n}\pi;$$
$$K_n = \max\{\operatorname{dist}\sigma(B), \operatorname{Re}\sigma(B + iC);\ B = B^*, C = C^*, \|C\| \leqslant 1\}.$$

Using the inequalities

$$\int_{(2s-1)\pi/(2n)}^{(2s+1)\pi/(2n)} \operatorname{ctg} x \leqslant \frac{\pi}{2n}\left(\operatorname{ctg}\frac{2s+1}{2n}\pi + \operatorname{ctg}\frac{2s-1}{2n}\pi\right),$$
$$\frac{\pi}{n}\operatorname{ctg}\frac{2s-1}{2n} \leqslant \int_{(s-1)\pi/n}^{(s+1)\pi/n}\operatorname{ctg} x \qquad (1 \leqslant s \leqslant [n/2]),$$

one may see that

$$\frac{1}{n}\operatorname{ctg}\frac{\pi}{2n} - \frac{2}{\pi}\ln\sin\frac{\pi}{2n} \leqslant K_n \leqslant \frac{2}{n}\operatorname{ctg}\frac{\pi}{2n} - \frac{2}{\pi}\ln\sin\frac{\pi}{n}$$

and that $K_n/\ln n \to 2/\pi$.

Infinite-dimensional case. Let H be a separable infinite-dimensional Hilbert space, and let A be a selfadjoint compact operator in H. λ_j^+, λ_j^-, $j = 1, 2, \ldots,$ denote the positive eigenvalues of A and $-A$, respectively, repeated according to multiplicity and arranged in decreasing order. If there are only n positive (negative) eigenvalues of A we set $\lambda_j^+ = 0$ ($\lambda_j^- = 0$) for $j > n$.

THEOREM 2.

$$\sup_{B = B^* \in L(H)} \operatorname{dist}(\sigma(B), \operatorname{Re} \sigma(B + iA)) = \frac{2}{\pi} \sum_{s=1} (\lambda_s^+ + \lambda_s^-)/(2s - 1).$$

PROOF. It follows from the Macaev theorem [3, Theorem III.4.2], or from Corollary 1 that "sup" is not less than the right-hand side. Thus it suffices to prove that if $B = B^* \in L(H)$ then

$$\operatorname{dist}(\sigma(B), \operatorname{Re} \sigma(B + iA)) < \frac{2}{\pi} \sum_{s=1} (\lambda_s^+ + \lambda_s^-)/(2s - 1). \tag{7}$$

It follows from the Weyl-von Neumann theorem [6, Theorem X.2.1] that there exists a compact selfadjoint operator K such that the operator $B + K$ has a pure point spectrum. Then there exists a sequence $\{P_n\}_1^\infty$ of orthogonal projections in H converging strongly to the identity operator such that P_n is n dimensional and commutes with $B + K$. Define the operators B_n, C_n by the formulas

$$B_n = (1 - P_n)B(1 - P_n) + P_n B P_n, \qquad C_n = B_n + iP_n A P_n.$$

Since $B_n - B = P_n(K - KP_n) + (K - P_n K)P_n$ it follows from the compactness of K that $\|B_n - B\| \to 0$. Since A is compact too we see that $\|C_n - (B + iA)\| \to 0$. The operators B_n, C_n are compact perturbations of B; therefore their essential spectra are identical. These facts and the perturbation theorems [6, Chapter IV, §3] imply that

$$\operatorname{dist}(\sigma(B), \sigma(B_n)) \to 0, \qquad \operatorname{dist}(\sigma(B + iA), \sigma(C_n)) \to 0. \tag{8}$$

Note further that $\sigma(B_n) = \sigma((1 - P_n)B|_{(1-P_n)H}) \cup \sigma(P_n B|_{P_n H})$, and $\sigma(C_n) = \sigma((1 - P_n)B|_{(1-P_n)H}) \cup \sigma(P_n(B + iA)|_{P_n H})$. Consequently,

$$\begin{aligned} \operatorname{dist}(\sigma(B_n), \operatorname{Re} \sigma(C_n)) &\leqslant \operatorname{dist}\big(\sigma(P_n B|_{P_n H}), \operatorname{Re} \sigma(P_n(B + iA)|_{P_n H})\big) \\ &\leqslant \frac{1}{n} \sum_{s=1}^{[n/2]} (\lambda_s^+ + \lambda_s^-) \operatorname{ctg} \frac{2s - 1}{2n} \pi < \frac{2}{\pi} \sum_{s=1}^{\infty} (\lambda_s^+ + \lambda_s^-)/(2s - 1), \end{aligned} \tag{9}$$

since the jth positive (negative) eigenvalue of $P_n A|_{P_n H}$, if it exists, is not greater (less) than λ_j^+ ($-\lambda_j^-$), and $\operatorname{ctg}((2s - 1)/2n)\pi < 2n/(2s - 1)\pi$. The desired inequality (7) now follows from (8) and (9).

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