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# ON STANDARD SUBGROUPS OF TYPE $^2E_6(2)$

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ABSTRACT. The purpose of this paper is to close one of the last gaps in the classification of finite simple groups containing a standard subgroup. We prove that a simple group containing a standard subgroup of type $^2E_6(2)$ has to be isomorphic to $F_2$, the baby monster.

One of the remaining standard form problems is the classification of finite groups containing a standard subgroup of type $^2E_6(2)$. A quasi-simple group $A$ is said to be a standard subgroup in a group $G$ provided

(i) $N_G(A) = N_G(C_G(A))$,
(ii) $|C_G(A)|$ is even,
(iii) $|C_G(A) \cap C_G(A)^g|$ is odd for all $g \in G - N_G(A)$,
(iv) $[A, A^g] \neq 1$ for all $g \in G$.

The purpose of this paper is to handle the case $A/Z(A) \cong {}^2E_6(2)$ and $2|\,|Z(A)|$. Further we may assume $m_2(C_G(A)) = 1$. Otherwise a result due to M. Aschbacher and G. Seitz [2] yields $A \trianglelefteq G$. Furthermore a Sylow 2-subgroup of $C_G(A)$ is cyclic. Otherwise the classical involution theorem due to M. Aschbacher [1] yields $A \trianglelefteq G$. The case $A/Z(A) \cong {}^2E_6(2)$ and $|Z(A)|$ odd has been treated by G. Seitz in [8]. In this paper we prove

THEOREM. *Let $G$ be a finite group, $O(G) = 1$, and $A$ a standard subgroup in $G$ such that $A/Z(A) \cong {}^2E_6(2)$ and $|Z(A)|$ is even. Then $\langle A^G \rangle = A$ or $\langle A^G \rangle \cong F_2$, the baby monster.*

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For the properties of $^2E_6(2)$ used in this paper see [11] and [10]. For the remainder suppose $A \ntrianglelefteq G$.

## 1. Preliminary results.

(1.1) LEMMA. *The group $X = {}^2E_6(2)$ possesses only one class of elements $\omega$ of order five, $C_X(\omega) \cong Z_5 \times A_8$.*

PROOF. [11, Lemmas (6.10) and (6.2)].

(1.2) LEMMA. *Let $\nu$ be an element of order 11 in $X = {}^2E_6(2)$. Then $|N_X(\langle \nu \rangle)| = 110$ and $|C_X(\nu)| = 22$.*

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PROOF. [11, Lemmas (7.8) and (6.5)].

(1.3) LEMMA. *Let $z$ be a 2-central involution in $X = {}^2E_6(2)$. Then*
(i) *$C_X(z)$ is an extension of an extraspecial 2-group $Q$ of width* 10 *with $PSU_6(2)$.*
(ii) *All involutions of $X$ are conjugated in $X$ to elements of $Q$.*
(iii) *Let $T$ be a Sylow 2-subgroup of $C_X(z)$. Then $J(T/\langle z\rangle) = Q/\langle z\rangle$.*
(iv) *$C_T(z)/Q$ contains exactly one elementary abelian subgroup of order $2^9$.*

PROOF. (i) follows from [10, Lemma 2]; (ii) follows from [10, Lemma 3]. An easy computation using the tables given in [11, pp. 502–505] yields (iii) and (iv).

(1.4) LEMMA. *Let $X$ be a 2-fold covering group of ${}^2E_6(2)$. If $x \in X$ and $x^2 \in Z(X)$ then $x^2 = 1$.*

PROOF. [11, p. 503].

(1.5) LEMMA. *Let $G$ be a finite group containing an involution $d$ such that $C_G(d)/O(C_G(d))$ is a 2-fold covering of ${}^2E_6(2)$. Let $S \in \mathrm{Syl}_2(C_G(d))$. Then $Z(S)$ is elementary abelian of order* 4. *Let $B$ be a Sylow 2-subgroup of $O_{2',2}(C_G(Z(S)))$ and $g$ a 2-element normalizing $B$ and acting trivially on $C_G(Z(S))/O_{2',2}(C_G(Z(S)))$. Then $[g, d] = 1$ or*
(i) *$\langle B, g\rangle = C$ is extraspecial of width* 11 *and*
(ii) $|N_G(C)/C_G(C)C| = 2^{17} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$.

PROOF. $|Z(S)| = 4$ follows from Lemmas (1.3) and (1.4). As in the proof of [11, Lemma (4.4)] the existence of a field automorphism of ${}^2E_6(2)$ is not used. Application of [11, Lemma (4.4)] proves (i). The same argument is possible for [11, Lemmas (4.6) and (4.7)]. This yields (ii).

## 2. Proof of the theorem.

(2.1) LEMMA. *A Sylow 2-subgroup of $C_G(A)$ is of order* 2.

PROOF. Let $T_1 \in \mathrm{Syl}_2(A)$. Then $Z(T_1) = \langle z, d\rangle$ is of order 4, $\langle d\rangle \in \mathrm{Syl}_2(Z(A))$. Let $Y$ be the preimage of $J(T_1/Z(T_1))$. Then $Y' = \langle z\rangle$, by Lemma (1.3). Let $T \in \mathrm{Syl}_2(N_G(A))$. Then $Z(T) = \langle z, S\rangle$, $S < C_G(A)$. Furthermore $\langle z\rangle$char $T$. Let $S_1 = T \cap C_G(A)$ and $Y_1$ the preimage of $J(T/Z(T))$. Then $Z(Y_1) = \langle z, S_1\rangle$.

Assume $|S_1| > 4$. Then $T \in \mathrm{Syl}_2(G)$. By Lemmas (1.3)(ii) and (1.4) each involution of $AC_G(A)$ is conjugate in $A$ to an involution in $Y_1$. Let $y \in Y_1$, $d \neq y \sim d$ in $G$. Then $|Y_1 : C_{Y_1}(y)| = 2$. Let $T_2 = C_T(y) \in \mathrm{Syl}_2(C_{N_G(A)}(y))$ and $T_3 \subseteq C_G(y)$, $|T_3 : T_2| = 2$. As $\Omega_1(\phi(Z(C_{Y_1}(y)))) = \langle d\rangle$, $C_{Y_1}(y) \trianglelefteq T_3$. Choose $t \in T_3 - T_2$. As $y \sim yz$ and $d \nsim dz$ in $G$ we get $z^t \neq z$, otherwise $\langle z, y\rangle = \Omega_1(Z(T_3)) \sim \Omega_1(Z(T)) = \langle z, d\rangle$. Hence $C_{Y_1}(y) \cap C_{Y_1}(y)^t$ has to be elementary abelian. But this yields that $C_{Y_1}(y)^t C_{Y_1}(y)/C_{Y_1}(y)$ is abelian of rank at least 10. This contradicts Lemma (1.3)(iv) and the structure of $\mathrm{Out}({}^2E_6(2))$ [9]. Thus $d^G \cap AC_G(A) = d$. Let $y$ be an involution in $N_G(A) - C_G(A)A$. Then there is a fours group $V$ contained in $A$ such that $Vy \subseteq y^G$. Let $g \in G$ with $y^g = d$. Then $V^g \cap AC_G(A) \neq 1$. But this contradicts $d^G \cap AC_G(A) = d$. Thus $d^G \cap N_G(A) = d$. Now application of [3] yields the contradiction $A \trianglelefteq G$.

(2.2) LEMMA. $|G : N_G(A)|$ *is even.*

PROOF. Suppose $|G : N_G(A)|$ to be odd. Let $T \in \mathrm{Syl}_2(N_G(A))$. Then $Z(T) = \langle d, z\rangle$, $\langle d\rangle \in \mathrm{Syl}_2(Z(A))$. Let $E$ be the preimage of $J(T/Z(T))$. By Lemma (1.3) we have that $E$ is the direct product of $\langle d\rangle$ with an extraspecial group of width 10. Furthermore $E' = \langle z\rangle$. By Lemma (1.3)(ii) every involution of $A$ is conjugate in $A$ to an involution of $E$. Let $y \neq d$ be an involution in $E$ such that $y \sim d$ in $G$. As $d \nsim z \nsim dz \nsim d$, $y \sim yz$ in $G$. Further $|E : C_E(y)| = 2$. Set $T_1 = C_T(y)$. We may assume $T_1 \in \mathrm{Syl}_2(C_{N_G(A)}(y))$. Let $T_2 \leqslant C_G(y)$, $|T_2 : T_1| = 2$ and $x \in T_2 - T_1$. Then $z^x \neq z$, otherwise $\langle z, y\rangle \sim \langle z, d\rangle$. Hence $C_E(y)^x \neq C_E(y)$. Thus $|C_E(y)^x C_E(y)/C_E(y)| \geqslant 2^9$. Now Lemma (1.3)(iv) yields that $C_E(y)^x C_E(y)/C_E(y)$ is uniquely determined. But then (see [7, Lemma 1]) $C_E(C_E(y)^x)$ contains only involutions conjugate to $d$, $z$ or $dz$ in $A$. This is a contradiction. Hence $d^G \cap C_G(A)A = d$.

Suppose $y \in N_G(A) - A$, $y \sim d$ in $G$. Then $A$ contains a fours group $V$ such that $Vy \leqslant y^G$. But this contradicts $d^G \cap C_G(A)A = d$. Thus $d^G \cap N_G(A) = d$. Now the application of [3] yields the contradiction $A \trianglelefteq G$.

(2.3) LEMMA. *There are involutions in $N_G(A)$ acting as field automorphisms on $A/Z(A)$. In particular $|N_G(A) : C_G(A)A| = 2$.*

PROOF. According to Lemma (2.2) $d \sim dz$ in $G$. Thus there is a 2-element $g$ in $N_G(\langle d, z\rangle)$ with $d^g = dz$. Suppose $N_G(A) = C_G(A)A$. By Lemma (1.3)(i) $N_{N_G(A)}(\langle d, z\rangle)/O_{2,2'}(N_{N_G(A)}(\langle d, z\rangle)) \cong PSU_6(2)$.

Assume

$$[g, N_{N_G(A)}(\langle d, z\rangle)] \subseteq O_{2,2'}(N_{N_G(A)}(\langle d, z\rangle)).$$

Set $C = \langle O_2(C_G(\langle d, z\rangle)), g\rangle$. Then Lemma (1.5) yields that $C$ is extraspecial of width 11 and $|N_G(C)/C_G(C)C| = 2^{17} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$. Let $t$ be a 2-central involution in $C_G(\langle d, z\rangle)/O_2(C_G(\langle d, z\rangle))$. Set $\overline{N_G(C)} = N_G(C)/C_G(C)C$ and $\tilde{C} = C/\langle z\rangle$. Now $\bar{X} = O_2(C_{\overline{N_G(C) \cap N_G(A)}}(\bar{t}))$ is extraspecial of width 4. Furthermore $C_{\overline{N_G(C) \cap N_G(A)}}(\bar{t})/\bar{X} \cong U_4(2)$; Lemma (1.3)(i). The group $\bar{X}$ centralizes in $\tilde{C}$ a subgroup of order $2^7$, see [11, Lemma (5.1)]. As $C_{\overline{N_G(C) \cap N_G(A)}}(\bar{t})$ induces on $(\tilde{Y}/\langle\tilde{d}\rangle)^{\#}$ two orbits of length 27 and 36, we get that $\tilde{d}$ possesses exactly 28 conjugates under the action of $N_{\overline{N_G(C)}}(\bar{X})$. Thus $N_{\overline{N_G(C)}}(\bar{X})/\bar{X}$ is of order $2^8 \cdot 3^4 \cdot 5 \cdot 7$. As $U_4(2)$ is involved we get that this group has to be simple. Checking the list of groups in [5] we get a contradiction. Thus we have that $g$ induces an outer automorphism on $N_{N_G(A)}(\langle d, z\rangle)/O_{2,2'}(N_{N_G(A)}(\langle d, z\rangle))$. Then by Lemma (1.3)(i) $O_2(N_{N_G(A)}(\langle d, z\rangle))/\langle z\rangle$ is the unique elementary abelian subgroup of order $2^{21}$ in a Sylow 2-subgroup of $N_G(\langle d, z\rangle)/\langle z\rangle$. Hence $d^{C_G(z)} \cap O_2(N_{N_G(A)}(\langle d, z\rangle)) = \{d, dz\}$.

Assume now $O(N_G(A)) \neq 1$. Let $T$ be a Sylow 2-subgroup of $A$ containing $z$. Then $C_G(T) = O(N_G(A))\langle z, d\rangle$. Thus $d \sim dz$ in $N_G(O(N_G(A)))$. Now by induction we get that $AO(N_G(O(N_G(A))))$ is normal in $N_G(O(N_G(A)))$. But this is a contradiction. Thus $O(N_G(A)) = 1$.

Let $\omega$ be an element of order five in $C_G(z) \cap N_G(A)$. By Lemma (1.1) we may assume $d \sim dz$ in $C_G(\omega)$. By Lemmas (1.1) and (1.4) we have $C_A(\omega)/\langle\omega\rangle \cong \langle d\rangle \times A_8$. Let $S$ be a Sylow 2-subgroup of $C_A(\omega)$. Then $S$ contains an elementary abelian subgroup $B$ of order 32. Because of $d^{C_G(z)} \cap O_2(N_A(\langle d, z\rangle)) = \{d, dz\}$ we get that $d$ has exactly 10 conjugates under $N_{C_G(\omega)}(B)$. As $A_6$ contains no subgroup isomorphic to $\Sigma_3 \times \Sigma_3$ we get $O_3(N_{C_G(\omega)}(B)/C_{C_G(\omega)}(B))$ is nontrivial. But then we get the contradiction that $d$ is weakly closed in $B$ with respect to $N_{C_G(\omega)}(B)$.

Hence we have shown that $|N_G(A): C_G(A)A| = 2$. Then we may assume that $[g, N_{N_G(A)}(\langle d, z\rangle)] \subseteq O_{2,2'}(N_{N_G(A)}(\langle d, z\rangle))$, as $\mathrm{Out}(PSU_6(2)) \cong \Sigma_3$, see **[9]**. By Lemma (1.2) we may assume that $g$ centralizes an element $\nu$ in $N_{N_G(A)}(\langle d, z\rangle)$ with $\nu^{11} \in O(C_G(A))$. By Lemma (1.2) we get that a Sylow 2-subgroup $S_1$ of $N_G(\langle\nu\rangle) \cap N_{N_G(A)}(\langle d, z\rangle)$ is of order eight. Clearly $S_1$ is abelian. We may assume that $g$ normalizes $S_1$. Thus $d \notin \phi(S_1)$. Then $S_1$ has to be elementary abelian. Thus there is an involution in $N_G(A) - C_G(A)$ normalizing a subgroup of order 11 in $A$. Now the structure of $\mathrm{Aut}({}^2E_6(2))$ **[9]** yields that this involution induces a field automorphism on $A/Z(A)$.

(2.4) LEMMA. *We have* $\langle A^G\rangle \cong F_2$.

PROOF. By Lemma (2.3) and **[11]** it is enough to show $O(N_G(A)) = 1$. Suppose $K = O(N_G(A)) \neq 1$. Let $C = F^*(C_A(\langle d, z\rangle))$. As $d \sim dz$ in $G$, $d \sim dz$ in $N_G(C)$. We have $C_G(C) = K\langle d, z\rangle$. Thus $d \sim dz$ in $N_G(K)$. As $A < N_G(K)$ we get by induction $N_G(K)/O(N_G(K)) \cong F_2$ and $[K, \langle A^{N_G(K)}\rangle] = 1$. Let $Y$ be a Sylow 2-subgroup of $F^*(C_{N_G(K)}(z))$. Then $Y$ is extraspecial of width 11. The conjugacy classes of involutions in $N_{N_G(K)}(Y) - Y$ are listed in **[11**, Table VI**]**. Let $x$ be such an involution. Then it is easy to see that $C_{N_{N_G(K)}(Y)}(T) = Z(T) \times K$, for $T \in \mathrm{Syl}_2(C_{N_{N_G(K)}(Y)}(x))$. Thus $d \not\sim x$ in $C_G(z)$. Thus the weak closure of $d$ in $N_G(Y)$ with respect to $C_G(z)$ is contained in $Y$. Then $Y$ is strongly closed in $N_G(Y)$ with respect to $C_G(z)$. Application of **[4]** yields now $C_G(z) \subseteq N_G(K)$. Now the structure of centralizers of involutions in $N_G(K)$ yields that $N_G(K)$ controls $G$-fusion of 2-central involutions in $N_G(K)$. But then $G = N_G(K)$ by Holt's theorem **[6]**. This contradicts $O(G) = 1$. The lemma is proved.

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