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# FINITELY GENERATED NON-HOPFIAN GROUPS

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ABSTRACT. We discuss finitely generated groups that are badly non-Hopfian. Given any countable group $L$, we construct a finitely generated group $G = K \times H$ where $H$ is isomorphic to $G$ and $L$ is a two-step subnormal subgroup of $K$.

## 1.

1.1. Following P. Neumann, we call a group $G$ *weakly Hopfian* if $G = K \rtimes H$ and $H \simeq G$ imply $K = \{1\}$. In [3] he investigated weak Hopficity in finitely generated solvable groups and proved that every finitely generated nilpotent-by-nilpotent group is weakly Hopfian but there exists a two generator solvable group of length three that is not weakly Hopfian. These results have motivated the investigations here. We call a group $G$ *subnormally Hopfian* if $G = K \rtimes H$, $H$ a subnormal subgroup of $G$ and $H \simeq G$ imply $K = \{1\}$. $G$ is *directly* Hopfian if $G = K \times H$ and $H \simeq G$ imply $K = \{1\}$. We write $G = K \rtimes H$ to mean that $G = KH$, $K \lhd G$ and $K \cap H = \{1\}$.

1.2. It turns out that if $G$ is a finitely generated group that is not subnormally Hopfian then $G/B$ is not directly Hopfian, where $B$ is the group generated by all abelian subnormal subgroups of $G$–the Baer radical of $G$. We shall discuss only finitely generated groups that are not subnormally Hopfian. Theorem 1 gives some properties of such a group $G$. These properties imply, in particular, that $G$ cannot be solvable. Theorem 2 deals with the result mentioned at the beginning of this paragraph. Theorems 3 and 4 deal with the embedding problems. Simple constructions and use of the beautiful results of P. Hall in [1] show that any countable group $L$ can be subnormally embedded in a finitely generated group that is not directly Hopfian. We have not tried to pursue this line to obtain results similar to those of J. M. Tyrer-Jones in [4].

## 2.

2.1. For a given group $G$ and positive integer $m$, $\gamma_m(G)$ will denote the $m$th term of the lower central series of $G$. Thus $\gamma_1(G) = G$ and $\gamma_{m+1}(G) = [\gamma_m(G), G]$. Occasionally we shall write $G'$ to denote $\gamma_2(G)$. We shall also use the symbol $\mathfrak{B}$ for the class of groups generated by abelian subnormal sufbgroups, $\mathfrak{A}$ for abelian groups, $\mathfrak{N}$ for nilpotent groups, $\mathfrak{G}$ for finitely generated groups, max-$n$ for groups satisfying the maximal condition for normal subgroups and $\mathfrak{A}\mathfrak{N}$ for abelian-by-nilpotent groups.

2.2. *Statements and proofs of the theorems.*

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THEOREM 1. *If* $G = K \rtimes H \in \mathfrak{G}$, $H \simeq G$ *and* $H$ *subnormal in* $G$ *then the following hold.*

(i) $\gamma_m(H)$ *is normal in* $G$ *for some positive integer* $m$.

(ii) $K = K'$.

(iii) $K \cap M \in \mathfrak{B}$ *where* $M = H^G = \langle H^g;\ g \in G\rangle$.

PROOF. Since $H$ is subnormal in $G$ the $n$-fold commutator group $[K, H, \ldots, H] < K \cap H = \{1\}$. Thus $[\gamma_m(H), K] = \{1\}$ for some positive integer $m$ [2, Theorem 3.8]. This gives (i).

Since $\mathfrak{G} \cap \mathfrak{A}\mathfrak{N} \subset$ max-$n \subset$ class of Hopfian groups, $K < \gamma_2(\gamma_m(G))$. Therefore $\gamma_m(G) = K \times \gamma_m(H)$ and $\gamma_2(\gamma_m(G)) = K' \times \gamma_2(\gamma_m(H)) > K$. Thus $K = K'$. Let $*$ denote the natural map of $G$ onto $G/\gamma_m(H) = G^*$. Then $K \simeq K^*$ and $H^*$ is nilpotent of class $m$ and subnormal in $G^*$. Let $B^*$ be the Baer radical of $G^*$. Then $H^* < B^* \lhd G^*$. Thus $M^* < B^*$ where $M = \langle H^g;\ g \in G\rangle$. Thus $M^* \cap K^* < B^*$. But $M^* \cap K^* \simeq M \cap K$. Hence $M \cap K \in \mathfrak{B}$.

THEOREM 2. *Suppose* $G = K \rtimes H \in \mathfrak{G}$, $H$ *a subnormal subgroup of* $G$ *and* $H \simeq G$. *Let* $*$ *be the natural map of* $G$ *onto* $G/B = G^*$ *where* $B$ *is the Baer radical of* $G$. *Then* $G^* = K^* \times H^*$, $H^* \simeq G^*$ *and* $K^* \neq \{1\}$ *if* $K \neq \{1\}$.

PROOF. Let $M = \langle H^g;\ g \in G\rangle$. By Theorem 1, $K = K'$ and $K \cap M \subseteq B$. Thus $K^*$ is a finitely generated perfect group. Since a nontrivial group cannot be perfect and nilpotent, $K \nsubseteq B$ unless $K = \{1\}$. That $H^* \lhd G^*$ follows from Theorem 1. Let $\theta$ be the isomorphism of $G$ onto $H$ then $\theta(B)$ is the Baer radical of $H$ and hence $\theta(B) = B \cap H$ and $G/B \simeq H/B \cap H \simeq H^*$. This completes the proof.

Suppose that $G = K \times H \in \mathfrak{G}$, $H \simeq G$ and $K \neq \{1\}$. Write $K_1 = K$, $H_1 = H$. Since $H_1 \simeq G$, $H_1 = K_2 \times H_2$ where $K_2 \simeq K_1$, $H_2 \simeq G$. More generally,

$$G = K_1 \times K_2 \times \cdots \times K_m \times H_m, \qquad H_m \simeq G, K_i \simeq K,$$

for all $i = 1, \ldots, m$ and $m = 1, 2, \ldots$. Since $G = \langle g_1, \ldots, g_n\rangle$ for a suitable choice of generators $g_i \in G$, $g_i = k_{i1}k_{i2} \cdots k_{im}h_{im}$ where $k_{ij} \in K_j$, $h_{im} \in H_m$. Moreover this representation is unique. Thus for all $i = 1, \ldots, n$, $j = 1, 2, \ldots$, $k_{ij} = w_{ij}(g_1, \ldots, g_n) = w_{ij}(k_{1j}, \ldots, k_{nj})$ and $w_{ij}(k_{1l}, \ldots, k_{nl}) = 1$ if $l \neq j$. Let $\phi_j$ be the isomorphism of $K_j$ onto $K$. Then $K = \langle k_{1j}\phi_j, \ldots, k_{nj}\phi_j\rangle$, $j = 1, 2, \ldots,$ and so

$$w_{il}(k_{1j}\phi_j, \ldots, k_{nj}\phi_j) = \begin{cases} 1, & \text{if } j \neq l, \\ k_{ij}, & \text{if } j = l. \end{cases} \tag{$*$}$$

This means that $K$ has infinitely many generating sets $S_j = \{k_{1j}\phi_j, \ldots, k_{nj}\phi_j\}$ and words $w_{il}$ satisfying $(*)$. These are nonisomorphic sets in the sense that no automorphism of $K$ can map $S_j$ onto $S_l$ as ordered sets. A simple way to construct such a group is to take $K$ to be a finitely generated simple group containing a wreath product $X$ wr $T$ where $X$ is any nonabelian group and $T = \langle t\rangle$ is infinite cyclic. Let $a, b \in X$ such that $[a, b] = c \neq 1$. Write $x_i$ to denote $t^{-i}xt^i$, $x \in X$. We can take $K$ to be a two-generator group as shown by Wilson in [5, p. 20], and we can take $X$ to be a nonabelian finite $p$-group so that $X$ wr $T$ is linear. Suppose that $K = \langle x, y\rangle$. Let $C$ be the cartesian product of copies $K_i$ of $K$, $i \in \mathbf{N}$; let $G$ be the subgroup of $C$ generated by $\underline{x} = (x, x, \ldots)$, $\underline{y} = (y, y, \ldots)$ and $\tau = (1, t, t^2, \ldots)$. Then the diagonal subgroup $\Delta$ of $C$ is contained in $G$ and if we denote by $\underline{g}$ the element $(g, g, \ldots)$ of $\Delta$, then $[\underline{a}^{\tau}, \underline{b}] = ([a, b], 1, 1, \ldots)$. Since $K$ is simple, $\bar{K}_1 \leqslant G$. Similarly $K_i$ and hence the direct product of $K_i$, $i \in \mathbf{N}$, is a subgroup of $G$. Let $H_1 = \langle \underline{x}', \underline{y}', \tau' \rangle$ where $\underline{x}' = (1, x, x, \ldots)$, $\underline{y}' = (1, y, y, \ldots)$ and $\tau' = \tau \underline{t}'^{-1} = (1, 1, t, t^2, \ldots)$. Then $H_1 < G$, $H_1 \simeq G$, and $G = K_1 \times H_1$. Thus we have shown

THEOREM 3. *There exists a three-generator group $G$ that is not directly Hopfian.*

Now let $K = \langle x, y \rangle$ be as above and let $K^* = \langle x^*, y^* \rangle$ be a copy of $K$. If $L$ is any countable group, then by Theorem A of [**1**], there exists a group $J = \langle K, K^* \rangle$ such that $L$ is a two-step subnormal subgroup of $J$. Let $C$ be the cartesian product of copies $J_i$, $i \in \mathbf{N}$, of $J$ and let $\Delta$ denote the diagonal subgroup of $C$. Then $\Delta = \langle \underline{x}, \underline{y}, \underline{x}^*, \underline{y}^* \rangle$ where $\underline{g} = (g, g, \ldots)$ for all $g \in J$. The subgroup $G = \langle \Delta, \tau, \tau^* \rangle$ where $\tau = (1, t, t^2, \ldots)$ and $\tau^* = (1, t^*, t^{*2}, \ldots)$ is the direct product of $J_1$ and $H_1$ where $H_1 = \langle \Delta', \tau', \tau^{*\prime} \rangle$. Here $\Delta'$ is generated by $\underline{g}' = (1, g, g, \ldots)$, $g \in J$, and $\tau' = \tau \underline{t}'^{-1} = (1, 1, t, t^2, \ldots)$. $\tau^{*\prime}$ is defined similarly. It is easy to see that $J_1 < G$ for $[\underline{a}^{\tau}, \underline{b}] = ([a, b], 1, 1, \ldots)$ and since $K$ is simple, $\langle [\underline{a}, \underline{b}]^{\Delta} \rangle < \{(k, 1, 1, \ldots); k \in K\}$. Similarly using $\underline{a}^*$, $\underline{b}^*$ and $\tau^*$ we get $G > \{(k^*, 1, 1, \ldots); k^* \in K^*\}$. Since $J = \langle K, K^* \rangle$, $G \geqslant J_1$. The rest follows easily from this and we have the following final result.

THEOREM 4. *Let $L$ be any countable group. Then there exists a six-generator group $G = K \times H$ such that $H \simeq G$ and $L$ is a two-step subnormal subgroup of $K$.*

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