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A UNIQUENESS CRITERION FOR ORDINARY DIFFERENTIAL EQUATIONS IN BANACH SPACES

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ABSTRACT. A uniqueness theorem for the Cauchy problem for ordinary differential equations in complex Banach spaces is given. This paper generalizes and extends a number of known results.

1. Introduction. We shall concern ourselves with the initial value problem

$$x'(t) = f(t, x(t)), \quad t \in (0, a), \qquad x(0) = x_0, \tag{1}$$

where f is a function which maps $(0, a) \times E$ into E, a Banach space. If $b \in (0, a]$, a solution of (1) in $[0, b)$ is a function x: $[0, b) \to E$, continuous in $[0, b)$, differentiable in $(0, b)$ which satisfies (1). This definition may be relaxed assuming that x is a solution in Carathéodory's sense (see Goldstein [**6**]).

Recently Medeiros [**10**] and Díaz and Weinacht [**4**] have studied uniqueness conditions for (1) in a complex Hilbert space. Their results have been extended by Goldstein [**6**], [**7**] to real or complex Banach spaces.

On the other hand, Nagumo's classic criterion has been modified by Bownds and Metcalf [**3**], Rogers [**11**], Gard [**5**] and Bernfeld, Driver and Lakshmikantham [**2**].

The present paper gives a uniqueness theorem which contains those of Goldstein as particular cases and extends to Banach spaces those of Gard and Rogers.

2. Preliminary results. Let E be a complex Banach space, with dual E^*. We denote by $\langle x, \psi \rangle$ the image of $x \in E$ by $\psi \in E^*$. For each $x \in E$ let $J(x)$ denote the (nonempty) set of all $\psi \in E^*$ such that

$$\langle x, \psi \rangle = \|x\|^2 = \|\psi\|^2.$$

We call J the duality mapping of E. If f is an operator we write its domain as $D(f)$. Let f: $D(f) \subset E \to E$; we say that f is dissipative if for x, y given in $D(f)$ there exists $\psi \in J(x - y)$ such that

$$\operatorname{Re}\langle f(x) - f(y), \psi \rangle \leq 0.$$

When E is a Hilbert space, J is the identity and then f is dissipative provided

$$\operatorname{Re}\langle f(x) - f(y), x - y \rangle \leq 0 \qquad \forall x, y \in D(f).$$

The following result, due to Kato [**9**, p. 510] will be used in the proof of the theorem.

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LEMMA. *Assume that u has a weak derivative $u'(s) \in E$ at $t = s$, and that $\|u(\cdot)\|$ is differentiable at $t = s$. Then*

$$\|u(s)\|\left[d\|u(t)\|/dt\right]_{t=s} = \mathrm{Re}\langle u'(s), \psi\rangle$$

for each $\psi \in J(u(s))$.

3. The uniqueness theorem.

THEOREM. *Let ϕ be a function continuous in $[0, a)$, differentiable in $(0, a)$, and such that $\phi(0) = 0$, $\phi(t) > 0$ for $t > 0$. Assume that $u(t)$, $v(t)$ are two solutions of (1) in $[0, b)$, $b \in (0, a)$ verifying*

(i) $\|f(t, u(t)) - f(t, v(t))\| = o[\phi'(t)]$.

Then, if

(ii) *$f(t, \cdot) - \phi'(t)I/\phi(t)$ is dissipative for all $t \in (0, b)$ it follows that $u(t) \equiv v(t)$.*

PROOF. First, note that (ii) is equivalent to the existence, for each pair x, $y \in D(f(t, \cdot))$, of a $\psi \in J(x - y)$ such that

$$\mathrm{Re}\langle f(t, x) - f(t, y), \psi\rangle \leqslant \phi'(t)\|x - y\|^2/\phi(t).$$

Let $u(t)$, $v(t)$ be two solutions of (1). We may assume, without loss of generality, that $u(t) \neq v(t)$ for $t \in (0, b)$. Define

$$m(t) = \begin{cases} \dfrac{1}{2}\left[\dfrac{\|u(t) - v(t)\|}{\phi(t)}\right]^2 & \text{if } t \in (0, b), \\ 0 & \text{if } t = 0. \end{cases}$$

For each $\xi \in E^*$, L'Hôpital's rule and (i) yield

$$\begin{aligned} \lim_{t\to 0+} \xi\left[\frac{u(t) - v(t)}{\phi(t)}\right] &= \lim_{t\to 0+} \frac{(\xi[u(t) - v(t)])'}{\phi'(t)} \\ &= \lim_{t\to 0+} \xi\left[\frac{f(t, u(t)) - f(t, v(t))}{\phi'(t)}\right] = 0 \end{aligned}$$

and therefore

$$\frac{u(t) - v(t)}{\phi(t)} \xrightarrow{w} 0.$$

Now, as a consequence of the Banach-Steinhaus theorem [**1**, p. 255], each weakly convergent sequence is bounded, and hence for any sequence $t_n \to 0+$ there exists $M > 0$ such that

$$\left\|\frac{u(t_n) - v(t_n)}{\phi(t_n)}\right\| \leqslant M, \quad \text{for } n = 1, 2, 3, \ldots. \tag{2}$$

We use Kato's lemma to obtain that for each $t \in (0, b)$ and each $\psi \in J(u(t) - v(t))$

$$\|u(t) - v(t)\|\frac{d}{dt}\|u(t) - v(t)\| = \mathrm{Re}\langle u'(t) - v'(t), \psi\rangle.$$

Apply again L'Hôpital's rule to get

$$\lim_{t\to 0+} m(t) = \frac{1}{2}\lim_{t\to 0+} \frac{\|u(t) - v(t)\|^2}{[\phi(t)]^2}$$
$$= \frac{1}{2}\lim_{t\to 0+} \frac{\|u(t) - v(t)\| \frac{d}{dt}\|u(t) - v(t)\|}{\phi(t)\phi'(t)}$$
$$= \frac{1}{2}\lim_{t\to 0+} \mathrm{Re}\left\langle \frac{u'(t) - v'(t)}{\phi'(t)}, \frac{\psi}{\phi(t)}\right\rangle$$
$$\leq \frac{1}{2}\lim_{t\to 0+} \left\|\frac{u'(t) - v'(t)}{\phi'(t)}\right\| \left\|\frac{u(t) - v(t)}{\phi(t)}\right\|.$$

In the last term the first factor tends to zero by (i) and upon comparison with (2) we see that $\lim_{t\to 0+} m(t) = 0$ and so $m(t)$ is continuous in $[0, b)$. Hence for $t > 0$, the mean value theorem implies there is an s in $(0, t)$ such that

$$0 < m(t) = tm'(s) = t\frac{1}{\phi(s)}\left(\mathrm{Re}\langle f(s, u(s)) - f(s, v(s)), \psi\rangle - \frac{\phi'(s)}{\phi(s)}\|u(s) - v(s)\|^2\right) \leq 0.$$

This is a contradiction and therefore $u(t) \equiv v(t)$ in $[0, b)$.

Some well-known theorems of uniqueness can be obtained as corollaries of the preceding theorem. For instance

COROLLARY 1 (GOLDSTEIN [7]). *Assume that for some $n \in N$, $f(t, \cdot) - nI/t$ is dissipative for each $t \in (0, b)$. Then, given $u_0, u_1, \ldots, u_n \in E$ there is at most a solution of* (1) *in* $[0, b)$ *such that $u^{(k)}(0)$ exists for $k = 0, 1, \ldots, n$ and $u^k(0) = u_k$, $k = 0, 1, \ldots, n$.*

PROOF. Take $\phi(t) = t^n$. Taylor expansions reveal that

$$u'(t) = \sum_{k=1}^{n} \frac{t^{k-1}u^{(k)}(0)}{(k-1)!} + o(t^{n-1}) = \sum_{k=1}^{n} \frac{t^{k-1}}{(k-1)!}u_k + o(t^{n-1}),$$
$$v'(t) = \sum_{k=1}^{n} \frac{t^{k-1}v^{(k)}(0)}{(k-1)!} + o(t^{n-1}) = \sum_{k=1}^{n} \frac{t^{k-1}}{(k-1)!}u_k + o(t^{n-1})$$

where u, v are solutions satisfying the conditions of the corollary. Clearly

$$\|f(t, u(t)) - f(t, v(t))\| = o(t^{n-1})$$

and the theorem applies.

COROLLARY 2. *Let ϕ satisfy the conditions of the theorem. Suppose also that*
(i) *$f(t, x) = h(t) + o[\phi'(t)]$ as $(t, x) \to (0+, x_0)$,*
(ii) *$f(t, \cdot) - \phi'(t)I/\phi(t)$ is dissipative for each $t \in (0, a)$.*
Then (1) *has at most one solution.*

This corollary extends a theorem given by Gard [5] for the case $E = R^n$. When $\phi(t) = t$, Nagumo's criterion is obtained, and when $\phi(t) = e^{-1/t}$ we arrive at Roger's result [**11**].

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