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HOLOMORPHIC MAPS THAT EXTEND TO AUTOMORPHISMS OF A BALL

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ABSTRACT. It is proved, under hypotheses that may be close to minimal, that certain types of biholomorphic maps of subregions of the unit ball in $\mathbf{C}^n$ have the extension property to which the title alludes.

Let B (or B_n, when necessary) denote the open unit ball of $\mathbf{C}^n$. Thus $z = (z_1, \ldots, z_n) \in B$ provided that $|z| < 1$, where $|z| = \langle z, z\rangle^{1/2}$ and $\langle z, w\rangle = \sum z_j\bar{w}_j$. An *automorphism* of B, i.e., a member of $\mathrm{Aut}(B)$, is, by definition, a holomorphic map of B onto B that is one-to-one, and whose inverse is therefore also holomorphic. The sphere that bounds B is denoted by S.

The following extension theorem will be proved.

THEOREM. *Assume that $n > 1$, and that*

(a) *Ω_1 and Ω_2 are connected open subsets of B,*

(b) *for $j = 1, 2$, Γ_j is an open subset of S such that $\Gamma_j \subset \partial\Omega_j$,*

(c) *F is a holomorphic one-to-one map of Ω_1 onto Ω_2, and*

(d) *there is a point $\alpha \in \Gamma_1$, not a limit point of $B \cap \partial\Omega_1$, and a sequence $\{a_i\}$ in Ω_1, converging to α, such that $\{F(a_i)\}$ converges to a point $\beta \in \Gamma_2$, not a limit point of $B \cap \partial\Omega_2$.*

Then there exists $\Phi \in \mathrm{Aut}(B)$ such that $\Phi(z) = F(z)$ for all $z \in \Omega_1$.

The relation of this theorem to earlier results will be discussed after its proof.

The proof will use the following well-known facts.

(I) *If F: $B_k \to B_n$ is holomorphic, and $F(0) = 0$, then $|F(z) \leq |z|$ for all $z \in B_k$, and the linear operator $F'(0)$ (the Fréchet derivative of F at 0) maps B_k into B_n.*

(II) *If, in addition, $k = n$, then the Jacobian JF of F satisfies $|(JF)(0)| \leq 1$; equality holds only when F is a unitary operator on $\mathbf{C}^n$.*

(III) *If $F \in \mathrm{Aut}(B)$ and $F(0) = 0$, then F is unitary.*

Here is a brief indication of how these are proved. For unit vectors u and v in $\mathbf{C}^k$ and $\mathbf{C}^n$, respectively, the classical Schwarz lemma applies to the function g defined by

$$g(\lambda) = \langle F(\lambda u), v\rangle, \qquad (\lambda \in \mathbf{C}, |\lambda| < 1). \tag{1}$$

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Thus $|g(\lambda)| < |\lambda|$ for all eligible u, v, which leads to $|F(z)| < |z|$, and $|g'(0)| < 1$, which completes (I), since

$$g'(0) = \langle F'(0)u, v\rangle. \tag{2}$$

Since (I) implies that no eigenvalue of $F'(0)$ exceeds 1 in absolute value, it follows that

$$|(JF)(0)| = |\det F'(0)| < 1. \tag{3}$$

If $|(JF)(0)| = 1$, then the linear operator $F'(0)$ preserves volume, and maps B into B, hence is a unitary operator U. From this it follows easily (by considering iterates of $U^{-1}F$) that $F = U$.

To prove (III), apply (II) to F as well as to F^{-1}.

The following lemma contains the essence of the proof of the theorem. To state it, we introduce the notation (for $z \in \mathbf{C}^n$)

$$D_z = \{\lambda z \colon \lambda \in \mathbf{C}, \lambda z \in B\}. \tag{4}$$

Thus, when $z \neq 0$, D_z is the disc that is the intersection with B of the complex line through 0 and z.

LEMMA. *Assume that*

(i) Ω_1 *and* Ω_2 *are connected open sets in* B,

(ii) $0 \in \Omega_1$, $0 \in \Omega_2$.

(iii) F *is a holomorphic one-to-one map of* Ω_1 *onto* Ω_2, *with* $F(0) = 0$, *and*

(iv) *there is a nonempty open set* $V \subset \Omega_1$, *such that* $D_z \subset \Omega_1$ *and* $D_{F(z)} \subset \Omega_2$ *for every* $z \in V$.

Then there is a unitary transformation U *on* $\mathbf{C}^n$ *such that* $F(z) = Uz$ *for all* $z \in \Omega_1$.

PROOF OF THE LEMMA. If $z \in V$, then D_z lies in the domain of F. Identifying D_z with B_1, we see from fact (I) (the case $k = 1$), that $|w| < |z|$, where $w = F(z)$. But D_w lies in the domain of F^{-1}, and the same argument shows that $|z| < |w|$. Thus $|F(z)|^2 = |z|^2$ for all $z \in V$. Both of these functions are real-analytic, hence they are equal in all of Ω_1. In particular, choosing $r > 0$ so small that $rB \subset \Omega_1$, we see that $|F(z)| = |z|$ for all $z \in rB$. An appropriately scaled version of fact (III) shows now that F is unitary.

PROOF OF THE THEOREM. Let $\{a_i\}$ be as in assumption (d), put $b_i = F(a_i)$, and choose $u_i \in S$, $v_i \in S$, so that

$$a_i = |a_i|u_i, \quad b_i = |b_i|v_i, \quad (i = 1, 2, 3, \dots). \tag{5}$$

The geometric information contained in (d) shows that there exists $t < 1$ such that, setting

$$E_t(\xi) = \{z \in B \colon t < \operatorname{Re}\langle z, \xi\rangle\}, \qquad (\xi \in S), \tag{6}$$

we have $a_i \in E_t(u_i) \subset \Omega_1$, and $b_i \in E_t(v_i) \subset \Omega_2$ for all sufficiently large i, say $i > i_0$.

If $a \in B \setminus \{0\}$, let P denote the orthogonal projection of $\mathbf{C}^n$ onto the one-dimensional subspace spanned by a, put $Q = I - P$, and define

$$\varphi_a(z) = \frac{a - Pz - (1 - |a|^2)^{1/2}Qz}{1 - \langle z, a\rangle}, \qquad (z \in \bar{B}). \tag{7}$$

Then (see [4], for instance) $\varphi_a \in \mathrm{Aut}(B)$ and $\varphi_a^{-1} = \varphi_a$. Define

$$G_i = \varphi_{b_i} \circ F \circ \varphi_{a_i}, \qquad (i > i_0). \tag{8}$$

Each G_i is a holomorphic one-to-one map of $\Omega_1^i = \varphi_{a_i}(\Omega_1)$ onto $\Omega_2^i = \varphi_{b_i}(\Omega_2)$, and $G_i(0) = 0$.

If $a = |a|\xi$, then $\langle Pz, \xi\rangle = \langle z, \xi\rangle\xi$, hence

$$\langle \varphi_a(z), \xi\rangle = (|a| - \langle z, \xi\rangle)/(1 - |a|\langle z, \xi\rangle). \tag{9}$$

If $t < |a|$, it follows that $\varphi_a(E_t(\xi))$ contains all $z \in B$ with

$$\mathrm{Re}\langle z, \xi\rangle < (|a| - t)/(1 - |a|t). \tag{10}$$

Since $|a_i| \to 1$ and $|b_i| \to 1$, and since the right side of (10) tends to 1 as $|a|$ tends to 1, there is a sequence $\{r_i\}$, $r_i < 1$, such that $r_i \to 1$ as $i \to \infty$, and such that

$$z \in B,\ \mathrm{Re}\langle z, u_i\rangle < r_i \text{ implies } z \in \Omega_1^i, \tag{11}$$

$$w \in B,\ \mathrm{Re}\langle z, v_i\rangle < r_i \text{ implies } w \in \Omega_2^i. \tag{12}$$

By (11), $r_iB \subset \Omega_1^i$, the domain of G_i. Since $G_i(0) = 0$, fact (II) gives $|(JG_i)(0)| \leqslant r_i^{-n}$. In the same way, (12) leads to $|(JG_i^{-1})(0)| < r_i^{-n}$, so that $|(JG_i)(0)| \geqslant r_i^n$. A normal family argument shows now that a subsequence of $\{G_i\}$ converges, uniformly on compact subsets of B, to a holomorphic map of B into B that fixes 0 and whose Jacobian at 0 has absolute value 1. By fact (II), this limit map is unitary. Call it U.

Let V_i be the set of all $p \in B$ such that

$$D_z \subset \Omega_1^i \quad \text{and} \quad D_{Uz} \subset \Omega_2^i \tag{13}$$

for all z in some neighborhood of p.

Now fix ε, $0 < \varepsilon < 1/10$. Using (11)–(13), we see that there is an index i, fixed from now on, such that

$$|G_i(z) - Uz| < \varepsilon \quad \text{whenever } |z| \leqslant 1 - \varepsilon, \tag{14}$$

and such that V_i contains a ball of radius 2ε, whose center p satisfies $|p| < 1 - 3\varepsilon$. To see in more detail that this can indeed be done, note that when r_i is sufficiently close to 1, there exists a large set of points $\xi \in S$ such that $|\langle \xi, u_i\rangle| < r_i$ and $|\langle \xi, U^{-1}v_i\rangle| < r_i$. For any such ξ, $D_\xi \subset \Omega_1^i$ and $D_{U\xi} \subset \Omega_2^i$, thus $\lambda\xi \in V_i$ if $0 < |\lambda| < 1$.

Thus $D_z \subset \Omega_1^i$ if $|z - p| < 2\varepsilon$, and $D_w \subset \Omega_2^i$ if $|w - Up| < 2\varepsilon$. If $|z - p| < \varepsilon$, and $w = G_i(z)$, it follows that $D_w \subset \Omega_2^i$ because

$$|w - Up| \leqslant |G_i(z) - Uz| + |z - p| < 2\varepsilon. \tag{15}$$

The lemma applies therefore to G_i and shows that G_i is (the restriction of) a unitary operator. Since (8) gives

$$F = \varphi_{b_i} \circ G_i \circ \varphi_{a_i}, \tag{16}$$

the theorem is proved.

REMARKS. (i) Let Ω be a connected open subset of B such that $\bar\Omega$ contains an open subset Γ of S. If F is a nonconstant C^1-map of $\bar\Omega$ into $\bar B$ that is holomorphic in Ω and carries Γ into S, then $F \in \mathrm{Aut}(B)$. This was proved by Pinčuk [6, p. 381], who extended an earlier version due to Alexander [1] in which C^∞ was assumed in place of C^1.

This Alexander-Pinčuk result is a fairly direct corollary of the present theorem. If $F \in C^1(\bar{\Omega})$ satisfies the Alexander-Pinčuk hypotheses, it is not hard to show (see Fornaess [3, p. 549] or Pinčuk [6, p. 378]) that JF vanishes at no point of Γ. The inverse function theorem implies then that the hypotheses of the present theorem hold.

(ii) In Alexander's proof [2] that every proper holomorphic map of B into B is in $\mathrm{Aut}(B)$ when $n > 1$, his appeal to Fefferman's theorem can be replaced by the one proved in the present paper. Consequently, there exists now a much more elementary proof of the proper mapping theorem for B.

(iii) It is quite possible that the present theorem remains true if B is replaced by strictly pseudoconvex domains with real-analytic boundaries (as Pinčuk did in the C^1-case [7]), but an entirely different proof would have to be found; Rosay [8] (strengthening a result of Wong [9]) proved that if some boundary point ξ of a bounded domain $\Omega \subset \mathbf{C}^n$ is a point of strict pseudoconvexity, and if there exist automorphisms T_k of Ω such that $\lim_{k\to\infty} T_k(p) = \xi$ for some $p \in \Omega$, then Ω is biholomorphically equivalent to B.

In other strictly pseudoconvex bounded domains there are thus insufficiently many automorphisms to imitate the proof that works in B.

(iv) If $\xi \in S$ and $\Omega = B \cap \{z: |\xi - z| < 1\}$; in other words, if $\Omega = B \cap (\xi + B)$, then the map $z \to \xi - z$ of Ω onto Ω demonstrates the relevance of the assumptions concerning the location of the points α and β in our theorem.

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