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TOTAL STABILITY FOR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. The basic idea of this work is to use Lyapunov functionals to show that for neutral functional differential equations, uniform asymptotic stability implies total stability.

1. Introduction. In discussing the existence of almost periodic solutions of retarded functional differential equations connecting with boundedness, there are two ways. The one is to assume a separation condition for bounded solutions and the other is to assume that an almost periodic system has a bounded solution with some kind of stability properties, uniform asymptotic stability, total stability and so on. In particular, the existence of a bounded totally stable solution implies the existence of an almost periodic solution; but even that the equation satisfies a local uniform Lipschitz condition, the existence of a uniformly asymptotically stable solution, does not imply the existence of an almost periodic solution **[4]**. For neutral equations, the above relationship between the existence of almost periodic solutions and some kind of stability is not well understood yet. It seems to be reasonable that if the operator D is not stable in the sense defined in **[1]**, then the results obtained for retarded equations can be extended to neutral equations. We analyse in the following the relationship between uniform asymptotic stability and total stability and show that if the equations satisfy a uniform Lipschitz condition and the D operator is stable then uniform asymptotic stability implies total stability.

2. Preliminary. Suppose $r > 0$ is a given real number, $R = (-\infty, \infty)$, E^n is a real or complex n-dimensional linear vector space with norm $|\cdot|$, $C = C([-r, 0], E^n)$ is the Banach space of continuous functions mapping the interval $[-r, 0]$ into E^n with the topology of uniform convergence given by the norm $\|\phi\| = \sup_{-r<\theta<0}|\phi(\theta)|$. If $\sigma \in R$, $A \geqslant 0$ and $x \in C([\sigma - r, \sigma + A], E^n)$ then for any $t \in [\sigma, \sigma + A]$ we let $x_t \in C$ be defined by $x_t(\theta) = x(t + \theta)$, $-r \leqslant \theta \leqslant 0$. If Ω is an open subset of $R \times C$ and $f, D\colon \Omega \to E^n$ are given continuous functions we say that the relation

$$\frac{d}{dt} D(t, x_t) = f(t, x_t) \tag{1}$$

is a functional differential equation.

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We say that (1) is a neutral functional differential equation if D is linear in ϕ, $D(t, \phi) = \phi(0) - g(t, \phi)$, where $g(t, \phi) = \int_{-r}^{0} d\mu(t, \theta)\phi(\theta)$, μ is an $n \times n$ matrix function of bounded variation for $\theta \in [-r, 0]$ and there is a continuous scalar function $l(s)$ strictly increasing, $l(0) = 0$ such that

$$\left|\int_{-s}^{0}[d_\theta\mu(t, \theta)]\phi(\theta)\right| \leqslant l(s) \sup_{-s \leqslant \theta \leqslant 0} |\phi(\theta)|, \qquad t \in R. \tag{2}$$

If f takes bounded sets of $R \times C$ into bounded sets we have continuous dependence of solutions, continuity and continuation of solutions to a maximal interval of existence of (1) [**2**].

3. Main results. Consider the system of functional differential equations of neutral type

$$\frac{d}{dt} D(t, x_t) = f(t, x_t), \qquad x_\sigma = \phi, \tag{3}$$

and the perturbed system

$$\frac{d}{dt} D(t, y_t) = f(t, y_t) + h(t, y_t) \tag{4}$$

where D: $[\tau, \infty) \times C \to E^n$ ($E^n = R^n$ or C^n) is linear continuous and f, h: $[\tau, \infty) \times C \to E^n$ are continuous and take bounded sets of $R \times C$ into bounded sets. We assume also that f and h satisfy a Lipschitz condition with respect to ϕ in a neighborhood of the origin uniformly in t for t in bounded sets $f(t, 0) = 0$, $h(t, 0) = 0$. Under these conditions, systems (3) and (4) have a unique solution $x_t(\sigma, \phi)$, $x_\sigma(\sigma, \phi) = \phi$ through (σ, ϕ).

DEFINITION 1. The zero solution of (1) is uniformly stable if for each $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that $\|\phi\| < \delta$ implies that $\|x_t(\sigma, \phi)\| \leqslant \varepsilon$ for every $t \geqslant \sigma \geqslant 0$. The zero solution of (1) is uniformly asymptotically stable if it is uniformly stable and there is $\delta > 0$ and for any $\eta > 0$ there exists $T = T(\eta)$ such that $\|\phi\| < \delta$ implies $|x_t(\sigma, \phi)| \leqslant \eta$ for $t \geqslant \sigma + T(\eta)$.

DEFINITION 2. We say that the operator D is uniformly stable if there exist constants $k \geqslant 0$, $M \geqslant 0$ such that the solution $x(\sigma, \phi, H)$ of $D(t, x_t) = D(\sigma, \phi) + H(t) - H(\sigma)$; $t \geqslant \sigma$, $x_\sigma = \phi$ satisfies

$$\|x_t(\sigma, \phi, H)\| \leqslant k\|\phi\| + \sup_{\sigma \leqslant \mu \leqslant t} |H(\mu) - H(\sigma)|, \qquad t \geqslant \sigma, \qquad [\mathbf{1}].$$

In what follows we assume that D is a uniformly stable operator. We will need the following lemma.

LEMMA. *Assume that $f(t, \phi)$ satisfies a Lipschitz condition with respect to ϕ in a neighborhood of the origin uniformly for $t \geqslant \sigma$ and $f(t, 0) = 0$. If $x_t(\sigma, \phi)$ and $y_t(\sigma, \psi)$ are solutions of* (1) *then there exist constants $k_0 > 0$, $L_1 > 0$ such that*

$$\|x_t(\sigma, \phi) - y_t(\sigma, \psi)\| \leqslant k_0 e^{L_1(t-\sigma)}\|\phi - \psi\|, \qquad t \geqslant \sigma.$$

PROOF. Let $x(t)$ and $y(t)$ be solutions of (1) with initial conditions ϕ and ψ respectively; then

$$|x(t) - y(t)|$$
$$\leqslant \left|\phi(0) - \psi(0) + g(\sigma, (\phi - \psi)) + g(t, x_t - y_t) + \int_\sigma^t [f(s, x_s) - f(s, y_s)]\,ds\right|.$$

Then there are constants L and $\alpha > 0$ such that $|x(t) - y(t)| \leqslant L\|\phi - \psi\|/2 + \|x_t - y_t\|/2$, for $\sigma \leqslant t \leqslant \sigma + \alpha$. Since we can take $L \geqslant 1$ it is easy to see that $\|x_t - y_t\| \leqslant L\|\phi - \psi\|/2 - \|x_t - y_t\|/2$ and then $\|x_t - y_t\| \leqslant L\|\phi - \psi\|$. By iterating this inequality a number of times we have for $t \in [\sigma + (n - 1)\alpha, \sigma + n\alpha]$ and $t \in [\sigma, \sigma + \tau]$

$$\|x_t(\sigma, \phi) - y_t(\sigma, \psi)\| \leqslant L^n\|\phi - \psi\|.$$

Let us show that there exist positive constants k_0 and L_1 such that

$$L^n\|\phi - \psi\| \leqslant k_0 e^{L_1(t-\sigma)}\|\phi - \psi\|, \qquad \sigma \leqslant t \leqslant \tau.$$

We have to show that there exist $k_0 > 0$, $L_1 > 0$, independent of n, such that $k_0 e^{L_1(n-1)\alpha} \geqslant L^n$ or $\log k_0 + L_1(n - 1)\alpha \geqslant n \log L$. We choose k_0 in such a way that $\log k_0 \leqslant L_1\alpha$; then $L_1\alpha + L_1(n - 1)\alpha \geqslant n \log L$ and $L_1 \geqslant (\log L)/\alpha$ satisfies the lemma. Hence $\|x_t - y_t\| \leqslant k_0 e^{L_1(t-\sigma)}\|\phi - \psi\|$, $\alpha \leqslant t < \infty$.

If V: $[\tau, \infty) \times C \to R$ is a continuous function we define the *derivative* $V(t, \phi)$ of V along the solutions of (1) by

$$\dot{V}(t, \phi) = \dot{V}_{(1)}(t, \phi) = \overline{\lim_{h \to 0_+}}\ \frac{1}{h}\left[V(t + h, x_{t+h}(t, \phi)) - V(t, \phi)\right].$$

The following theorem is a corollary of Theorem 1 of [3].

THEOREM 1. *If the zero solution of* (3) *is uniformly asymptotically stable then there are constants* $\delta_0 > 0$, $k = k(\delta_0) > 0$, $M > 0$ *and continuous nondecreasing, positive definite functions* $u(s)$, $c(s)$, $b(s)$, $w(s)$, $v(s)$, *for* $0 \leqslant s \leqslant \delta_0$, $u(0) = c(0) = b(0) = w(0) = v(0) = 0$ *and a Lyapunov functional* V: $[\sigma, \infty) \times C \to R^n$, *such that*

(i) $u(|D(t, \phi)|) \leqslant V(t, \phi)$,

(ii) $c(\|\phi\|) \leqslant V(t, \phi) \leqslant b(\|\phi\|)$,

(iii) $\dot{V}_{(4)}(t, \phi) \leqslant \dot{V}_{(3)}(t, \phi) + M|h(t, \phi)|$,

(iv) $\dot{V}(t, \phi) \leqslant -v(|D(t, \phi)|)$, $\dot{V}(t, \phi) \leqslant -w(\|\phi\|)$,

(v) $|V(t, \phi) - V(t, \psi)| \leqslant k\|\phi - \psi\|$, $t \geqslant \sigma$, $\phi, \psi \in C([-r, 0], E^n)$, $\|\phi\|, \|\psi\| \leqslant \delta_0$.

PROOF. The existence of V satisfying conditions (i), (ii), (iv) and (v) is shown in [3, Theorem 1] by virtue of Lemma 1 and condition (2). The derivative of V along the solutions of (4) is

$$\dot{V}_{(4)}(t, \phi) \leqslant \dot{V}_{(3)}(t, \phi) + \frac{k}{1 - l(s_0)}|h(t, \phi)|$$
$$= \dot{V}_{(3)}(t, \phi) + M|h(t, \phi)|, \qquad t \geqslant \sigma,\ \|\phi\| \leqslant \delta,$$

where l is the function defined in condition (2) with s_0 small enough in such a way that $0 < l(s_0) < 1$, and $k > 0$ is the constant in condition (v) that proves (iii).

4. Total stability.

DEFINITION 3. The solution $x \equiv 0$ of (3) is totally stable if for every $\varepsilon > 0$ there exists $\eta_1(\varepsilon) > 0$, $\eta_2(\varepsilon) > 0$ such that $\|\phi\| < \eta_1$ and $|h(t, \phi)| < \eta_2$ implies that $\|y_t(\sigma, \phi)\| < \varepsilon$, $t \geqslant \sigma$, where $y_t(\sigma, \phi)$ is a solution of (4).

THEOREM 2. *If the solution* $x \equiv 0$ *of* (3) *is uniformly asymptotically stable it is totally stable.*

PROOF. Assume that there is a $t_1 > \sigma$ such that $\|y_{t_1}(\sigma, \phi)\| \geqslant \varepsilon$. Choose $\eta_1 > 0$, $l > 0$ such that $b(\eta_1) \leqslant l < c(\varepsilon/2)$ where b and c are functions in (ii) in Theorem 1; then

$$V(t_1, y_{t_1}(\sigma, \phi)) \geqslant c\|y_{t_1}(\sigma, \phi)\| \geqslant c(\varepsilon) \geqslant c(\varepsilon/2) > l,$$

$$V(\sigma, y_\sigma(\sigma, \phi)) \leqslant b(\|\phi\|) \leqslant b(\eta_1) \leqslant l.$$

Since $V(t, \phi)$ is continuous there exists t_2, $\sigma < t_2 < t_1$, such that $V(t_2, y_{t_2}(\sigma, \phi)) = l$ and $V(t, y_t(\sigma, \phi)) > l$ for $t > t_2$. Hence $b(\|y_{t_2}(\sigma, \phi)\|) \geqslant V(t_2, y_{t_2}(\sigma, \phi)) \geqslant c(\|y_{t_2}(\sigma, \phi)\|)$.

If $\psi = y_{t_2}(\sigma, \phi)$ we have $b(\|\psi\|) \geqslant l \geqslant c(\|\psi\|)$ and $b(\eta_1) \leqslant l < c(\varepsilon/2)$ which implies that $\eta_1 \leqslant \|\psi\| < \varepsilon$. From the inequalities (iii) and (iv) in Theorem 1 and choosing $\eta_2 < w(\eta_1)/M$ we have

$$\dot{V}_{(4)}(t_2, \psi) \leqslant -w(\|\psi\|) + M\eta_2 < 0; \tag{5}$$

we have also that

$$\dot{V}_{(4)}(t_2, \psi) = \overline{\lim_{t \to 0^+}} \frac{1}{h}\left[V(t_2 + h, y_{t_2+h}(t_2, \psi)) - V(t_2, \psi)\right] \geqslant 0.$$

Since $V(t_2, y_{t_2}(\sigma, \phi)) = V(t_2, \psi) = l$ and $V(t, y_t(\sigma, \phi)) > l$ for $t > t_2$ this is a contradiction. Then $\|y_t(\sigma, \phi)\| < \varepsilon$ for $t \geqslant \sigma$ and the proof is complete.

EXAMPLE 1. If $a > 0$ and $|c| < 1$, the solution $x = 0$ of the equation

$$\frac{d}{dt}\left[x(t) + cx(t - r)\right] = -ax(t) \tag{6}$$

is totally stable.

PROOF. Since $|c| < 1$ it is well known [2] that the operator $D(\phi) = \phi(0) + c\phi(-r)$ is uniformly stable. It is proved in [**1**, Example 6.1] that the solution $x \equiv 0$ of (6) is uniformly asymptotically stable and from Theorem 2 it is totally stable.

EXAMPLE 2. The transmission line without loss with two differential elements in the terminals [**5a**], gives rise to a system of second order equations

$$\begin{aligned} C_1 \frac{d}{dt} D(t, x_t) &= -\frac{1}{z} x(t) - \frac{q}{z} x(t - r) - g(D(t, x_t)) - i(t), \\ L_1 \frac{d}{dt} i(t) &= -R_1 i(t) + D(t, x_t) \end{aligned} \tag{7}$$

where C_1, L_1, K_1, R_1, z are positive constants. If $|q| < 1$, $D(t, \phi) = \phi(0) - q\phi(-r)$ is uniformly stable and if there exists H such that

$$\inf_{|x| > H} \frac{g(x)}{x} = M > -\frac{1}{z} \frac{1 - |q|}{1 + |q|},$$

then the system above is totally stable.

PROOF. We consider the operator

$$\overline{D}(\phi, j) = \left(\sqrt{C_1}\,(\phi(0) - q\phi(-r)), \sqrt{L_1}\, j\right) = \left(\sqrt{C_1}\, D\phi, \sqrt{L_1}\, j\right).$$

Define

$$V(\phi, j) = \frac{1}{2}\left[\overline{D}(\phi, j)\right]^2 + \beta\int_{-r}^{0}\phi^2(\theta)\, d\theta$$

where $\beta = |q|/z$. Since $\frac{1}{2}|\overline{D}(\phi, j)|^2 = (C_1/2)(D\phi)^2 + (L_1/2)j^2$ we have

$$V(\phi, j) = (C_1/2)(D\phi)^2 + (L_1/2)j^2 + \beta\int_{-r}^{0}\phi^2(\theta)\, d\theta.$$

If $(x_t, i(t))$ is a solution of (7) then

$$\dot{V}(x_t, i(t)) \leq F - R_i i^2(t)$$

where

$$F = (Dx_t)(-1/z \cdot x(t) - q/z \cdot x(t - r))$$
$$+ \beta x^2(t) - \beta x^2(t - r) - (g(Dx_t)/Dx_t)(Dx_t)^2,$$

choose γ such that $M > \gamma$, $\gamma > (-1/z)(1 - |q|)/(1 + |q|)$.

Adding and subtracting $\gamma(Dx_t)^2$ in $\dot{V}(x_t, i(t))$ we obtain

$$\dot{V}(x_t, i(t)) \leq Q - (Dx_t)^2(g(Dx_t)/(Dx_t) - \gamma) - R_1 i^2(t),$$
$$Q = (Dx_t)(-1/z \cdot x(t) - q/z \cdot x(t - r)) + \beta x^2(t) - \beta x^2(t - r) - \gamma(Dx_t)^2.$$

Q is a negative definite quadratic form on $x(t)$ and $x(t - r)$; then

$$\dot{V}(x_t, i(t)) \leq -(Dx_t)^2(g(Dx_t)/(Dx_t) - \gamma) - R_1 i^2(t).$$

For $|Dx_t| \geq H$ from the hypotheses on g we have

$$\dot{V}(x_t, i(t)) \leq -(Dx_t)^2(M - \gamma) - R_1 i^2(t)$$
$$\leq -\alpha(Dx_t)^2 - R_1 i^2(t)$$

where $\alpha = M - \gamma > 0$. Now we seek A such that

$$-\alpha(Dx_t)^2 - R_1 i^2(t) \leq A\left|\overline{D}(x_t, i(t))\right|^2$$

for every Dx_t, $i(t)$. If $A = \alpha/2C_1 \leq R_1/L_1$ this condition is satisfied then $\dot{V}(x_t, i(t)) \leq -A|\overline{D}(x_t, i(t))|^2$ and the solution $x = 0$ of (3) is uniformly asymptotically stable and from Theorem 2 is totally stable.

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