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NOT EVERY d-SYMMETRIC OPERATOR IS GCR

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ABSTRACT. Let T be an element of $\mathfrak{B}(\mathcal{H})$, the algebra of bounded linear operators on the Hilbert space $\mathcal{H}$. The derivation induced by T is the map $\delta_T(X) = TX - XT$ from $\mathfrak{B}(\mathcal{H})$ into itself. T is *d-symmetric* if the norm closure of the range of δ_T, $\mathfrak{R}(\delta_T)^-$, is closed under taking adjoints. This paper answers the question of whether every d-symmetric operator is GCR by giving an example of an NGCR weighted shift that is also d-symmetric.

Let $\mathcal{H}$ be a complex Hilbert space and T an element of $\mathfrak{B}(\mathcal{H})$, the algebra of bounded linear operators from $\mathcal{H}$ into $\mathcal{H}$. The derivation induced by T is the mapping $\delta_T(X) = TX - XT$ from $\mathfrak{B}(\mathcal{H})$ into itself. T is said to be *d-symmetric* if the norm closure of the range of δ_T, $\mathfrak{R}(\delta_T)^-$, is closed under taking adjoints. Examples of d-symmetric operators include the normal operators and isometries.

In [**ABDW**] it is proved that a necessary and sufficient condition for T to be d-symmetric is that $TT^* - T^*T \in \mathcal{C}(T)$ where $\mathcal{C}(T) = \{C \in \mathfrak{B}(\mathcal{H}): C\mathfrak{B}(\mathcal{H}) + \mathfrak{B}(\mathcal{H})C \subseteq \mathfrak{R}(\delta_T)^-\}$. In the same paper the question is raised whether every d-symmetric operator is GCR. This paper answers that question in the negative by giving an example of a weighted shift $Te_i = \alpha_i e_{i+1}$, $i \in \mathbf{Z}$, that is both d-symmetric and NGCR. Recall that an operator T is GCR if every irreducible representation of $C^*(T)$, the C^*-algebra generated by T and the identity operator, contains the compact operators. T is NGCR if $C^*(T)$ contains no GCR two sided ideal [**A**]. If T is irreducible then T is NGCR if and only if $C^*(T)$ contains no nonzero compact operator [**A**].

LEMMA. *Let V be similar to T, say $SVS^{-1} = T$. Then T is d-symmetric if and only if $S^{-1}(TT^* - T^*T)S \in \mathcal{C}(V)$.*

PROOF. $\delta_T(SXS^{-1}) = SVXS^{-1} - SXVS^{-1} = S\delta_V(X)S^{-1}$. Hence $\mathfrak{R}(\delta_T)^- = S\mathfrak{R}(\delta_V)^-S^{-1}$ and it follows that $\mathcal{C}(V) = S^{-1}\mathcal{C}(T)S$. Thus $C = TT^* - T^*T \in \mathcal{C}(T)$ if and only if $S^{-1}CS \in \mathcal{C}(V)$. The lemma now follows from the result quoted above. $\square$

We now restrict our attention to weighted shifts. Recall that two bilateral shifts $Ve_i = \alpha_i e_{i+1}$ and $Tf_i = \beta_i f_{i+1}$ are similar if and only if there exist integer k and constant C so that $1/C \leqslant |(\alpha_k \alpha_{k+1} \cdots \alpha_{k+n})/(\beta_0 \beta_1 \cdots \beta_n)| \leqslant C$ uniformly for

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all $n \in \mathbf{Z}$ (see **[S]**). If we define $T_0 e_i = \beta_{i-k} e_{i+1}$ then T_0 is unitarily equivalent to T, T_0 is similar to V, and the similarity can be implemented by an operator that is diagonal with respect to $\{e_n\}$ (see **[S]**). The same results are true in the unilateral case with $k = 0$, $n \in \mathbf{N}$. This leads to the following.

COROLLARY. *Let V and T be similar (unilateral or bilateral) weighted shifts. Then T is d-symmetric if and only if $T_0 T_0^* - T_0^* T_0 \in \mathcal{C}(V)$.*

PROOF. Since d-symmetry is clearly preserved under unitary equivalence, T is d-symmetric if and only if T_0 is d-symmetric. T_0 is similar to V by means of a diagonal operator D. $T_0 T_0^* - T_0^* T_0$ is diagonal with respect to the same basis so $D^{-1}(T_0 T_0^* - T_0^* T_0) D = T_0 T_0^* - T_0^* T_0$. $\square$

REMARK. If S is an invertible operator that commutes with C, then it is not hard to show that $C \in \mathcal{C}(T)$ if and only if $CS \in \mathcal{C}(T)$. In particular, if $C = T_0 T_0^* - T_0^* T_0$ is a diagonal operator as in the corollary, then $C \in \mathcal{C}(T)$ if and only if $|C|$, the diagonal with diagonal entries the modulus of the corresponding entry in C, is in $\mathcal{C}(T)$. (It is not true in general that $|C| \in \mathcal{C}(T)$ implies $C \in \mathcal{C}(T)$.)

In **[ABDW]** it is shown that when T is d-symmetric, $\mathcal{C}(T)$ is the linear span of the positive elements in $\mathfrak{R}(\delta_T)^-$. This implies the following.

PROPOSITION. *If V is a d-symmetric weighted shift and T is a weighted shift similar to V, then T is d-symmetric if and only if $|T_0 T_0^* - T_0^* T_0| \in \mathfrak{R}(\delta_V)^-$.*

Before we proceed to the example, we need to state a result due to O'Donovan. In **[O]** he proves that a bilateral shift with nonzero weights $\{w(i)\}$ is NGCR if and only if there exists a sequence $n_k \to \infty$, such that $w(i + n_k) \to w(i)$ for $i \in \mathbf{Z}$.

EXAMPLE. Let T be the bilateral weighted shift with weights defined by

$$w(i) = \begin{cases} 1, & i \leqslant 0, \\ \frac{1}{2}, & i = 1, \\ 2, & i = 2, \\ 1, & 3^k \leqslant i < 2 \cdot 3^k, \\ w(i - 2 \cdot 3^k), & 2 \cdot 3^k \leqslant i < 3^{k+1}. \end{cases}$$

CLAIM I. T is NGCR.

PROOF. Let $n_k = 2 \cdot 3^k$. Fix $i \leqslant 0$. Then for $k > 1$ so that $3^k > |i|$, $w(i + n_k) = w(2 \cdot 3^k - |i|) = 1 = w(i)$.

Fix $i > 0$. Then for k so that $3^k > i$ we have $2 \cdot 3^k \leqslant i + 2 \cdot 3^k < 3^{k+1}$ so $w(i + n_k) = w(i + 2 \cdot 3^k) = w(i)$. In any case we have $w(i + n_k) \to w(i)$. $\square$

CLAIM II. T is similar to the bilateral shift $Ve_n = e_{n+1}$ and $T_0 = T$.

PROOF. An induction argument shows that if $w(k) = 2$ then $w(k - 1) = \frac{1}{2}$ and if $w(k) = \frac{1}{2}$ then $w(k + 1) = 2$. Since all other weights are 1 it follows that

$$\tfrac{1}{2} \leqslant |w(0) \cdot w(1) \cdot \; \cdots \; \cdot w(n)| \leqslant 2 \quad \text{for } n \in \mathbf{Z}. \quad \square$$

Matrix computations show that $D = |TT^* - T^*T|$ is the diagonal operator with the weights

$$d(i) = \begin{cases} 0, & i \leqslant 0, \\ \frac{3}{4}, & i = 1, \\ \frac{15}{4}, & i = 2, \\ 3, & i = 3, \\ 0, & 3^k < i \leqslant 2 \cdot 3^k, \\ d(i - 2 \cdot 3^k), & 2 \cdot 3^k < i \leqslant 3^{k+1}. \end{cases}$$

In order to show T is d-symmetric it is enough to show that $D = |TT^* - T^*T| \in \mathfrak{R}(\delta_V)^-$ by the proposition. As

$$\delta_V\left(-\sum_{j=0}^{n-1}\left(\frac{n-j}{n}\right)V^jDV^{*(j+1)}\right) = D - \frac{1}{n}\sum_{j=1}^{n} V^jDV^{*j},$$

we will be done if we show $3^{-k}\|\sum_{j=1}^{3^k} V^jDV^{*j}\| \to 0$ as $k \to \infty$.

Since conjugation by V shifts a diagonal operator one position down the diagonal, $\sum_{j=1}^{n} V^jDV^{*j}$ is also a diagonal operator and its weights are $d'(i) = \sum_{j=1}^{n} d(i-j) = \sum_{j=1}^{n} d(i-n+j)$. Thus it suffices to show that

$$\frac{1}{3^k}\sum_{j=1}^{3^k} d(i+j) \to 0 \quad \text{uniformly in } i \text{ as } k \to \infty.$$

CLAIM III. $\sum_{j=1}^{3^k} d(j) < 8 \cdot 2^k$.

PROOF. If $k = 1$, then $\sum_{j=1}^{3} d(j) = 15/2 < 8$. Assuming $\sum_{j=1}^{3^k} d(j) < 8 \cdot 2^k$ we see that

$$\sum_{j=1}^{3^{k+1}} d(j) = \sum_{j=1}^{3^k} d(j) + \sum_{j=3^k+1}^{2\cdot 3^k} d(j) + \sum_{j=2\cdot 3^k+1}^{3^{k+1}} d(j)$$
$$= 2\sum_{j=1}^{3^k} d(j) < 8 \cdot 2^{k+1}. \quad \square$$

CLAIM IV. $\sum_{j=1}^{3^l} d(i+j) < 8 \cdot 2^l$ for all $i \in \mathbf{Z}$.

PROOF. Suppose that $-\infty < i \leqslant 3^l$. Since $d(j) = 0$ for $j \leqslant 0$ and $3^l < j \leqslant 2 \cdot 3^l$,

$$\sum_{j=1}^{3^l} d(i+j) = \sum_{j=i+1}^{3^l+i} d(j) \leqslant \sum_{j=1}^{2\cdot 3^l} d(j)$$
$$= \sum_{j=1}^{3^l} d(j) < 8 \cdot 2^l$$

by Claim III.

Let $k \geqslant l$ and assume that $\sum_{j=1}^{3^l} d(i+j) < 8 \cdot 2^l$ for $i \leqslant 3^k$. Let $3^k < i \leqslant 3^{k+1}$ and consider

$$\sum_{j=1}^{3^l} d(i+j) = \sum_{j=i+1}^{i+3^l} d(j).$$

If $i + 3^l \leqslant 2 \cdot 3^k$ then the sum is zero since $d(j) = 0$ for $3^k < j \leqslant 2 \cdot 3^k$. For the same reason we can assume that the lower limit on the sum is at least $2 \cdot 3^k$. Since $d(j) = 0$ for $3^{k+1} < j \leqslant 3^{k+1} + 3^l < 2 \cdot 3^{k+1}$, we can also assume that the upper limit is at most 3^{k+1}. Hence $\sum_{j=i+1}^{i+3^l} d(j) = \sum_{j=n+1}^{m} d(j)$ with $2 \cdot 3^k \leqslant n \leqslant m \leqslant 3^{k+1}$ and $m - n \leqslant 3^l$. Let $m' = m - 2 \cdot 3^k$ and $n' = n - 2 \cdot 3^k$; then we see that

$$\sum_{j=n+1}^{m} d(j) = \sum_{j=n'+1}^{m'} d(j) \leqslant \sum_{j=n'+1}^{n'+3^l} d(j) < 8 \cdot 2^l$$

since $n' \leqslant 3^{k+1} - 2 \cdot 3^k = 3^k$. Hence

$$\sum_{j=1}^{3^l} d(i+j) < 8 \cdot 2^l \quad \text{for } i \leqslant 3^{k+1}. \quad \square$$

Thus we have shown that T is both NGCR and d-symmetric.

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