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COMPATIBLE RELATIONS OF MODULAR AND ORTHOMODULAR LATTICES

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ABSTRACT. Let L be a modular lattice of finite length. L is a projective geometry if and only if L has only trivial tolerances.

A binary relation θ is called a tolerance of an algebra $\mathfrak{A} = (A, \Omega)$ if θ is reflexive, symmetric and compatible with the operations of $\mathfrak{A}$. The tolerances $D = \{(a, a)|a \in A\}$ and A^2 are called the trivial tolerances of $\mathfrak{A}$. Obviously every congruence relation of $\mathfrak{A}$ is also a tolerance of $\mathfrak{A}$. If L is a lattice then we consider $R = \{(a, b)|a, b \in A, a \leqslant b\}$ which is also a compatible relation of L. The lattice of the subalgebras ρ with $D \subseteq \rho \subseteq R$ will be denoted by S and the meet operation of S by $\cap$.

These different kinds of relations were already studied by Hashimoto [4] and by Grätzer and E. T. Schmidt, [7], [6]. The following theorem is a generalization of a result of G. Grätzer and E. T. Schmidt concerning congruence relations.

THEOREM 1. *Let T be the lattice of tolerances of L and S the lattice of subalgebras ρ with $D \subseteq \rho \subseteq R$. Then T and S are isomorphic.*

PROOF. We consider the order-preserving function t: $T \to S$ defined by $t(\eta) = \eta \cap R$, $\eta \in T$, and furthermore s: $S \to T$ defined by $s(\rho) = \xi$, $\rho \in S$, in the following way. $(a, b) \in \xi$ iff $(a \wedge b, b) \in \rho$, $(a \wedge b, a) \in \rho$, $(a, a \vee b) \in \rho$ and $(b, a \vee b) \in \rho$. It is clear that ξ is reflexive and symmetric. If $(a_1, b_1) \in \xi$ and $(a_2, b_2) \in \xi$ then $(a_1 \wedge b_1, b_1) \in \rho$ and $(a_2 \wedge b_2, b_2) \in \rho$ and therefore $((a_1 \wedge b_1) \vee (a_2 \wedge b_2); b_1 \vee b_2) \in \rho$. As $a_1 \vee a_2 \geqslant (a_1 \wedge b_1) \vee (a_2 \wedge b_2)$ and $b_1 \vee b_2 \geqslant (a_1 \wedge b_1) \vee (a_2 \wedge b_2)$ we have $((a_1 \vee a_2) \wedge (b_1 \vee b_2), b_1 \vee b_2) \in \rho$. Similarly we prove the three other conditions and have $(a_1 \vee a_2, b_1 \vee b_2) \in \xi$. In the same way we can show that ξ is compatible with the operation $\wedge$. The function s is also order-preserving. We have $t \circ s(\rho) = t(\xi) = \xi \cap R$. If $(c, d) \in \xi \cap R$ then we have $(c, d) = (c \wedge d, d) \in \rho$. If $(a, b) \in \rho$ then $(a, b) \in R$ and $(a \wedge b, b) \in \rho$, $(a \wedge b, a) \in \rho$, $(a, a \vee b) \in \rho$ and $(b, a \vee b) \in \rho$ and therefore $(a, b) \in \xi \cap R$. We have $t \circ s = 1_S$ and $s \circ t = 1_T$ is proved similarly.

THEOREM 2. *Let L be an orthomodular lattice. A binary relation θ of L is a congruence relation if and only if θ is reflexive, symmetric and compatible with join and meet.*

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PROOF. As L is relatively complemented θ is a lattice congruence of L [4], [7]. It remains to show that from $a\,\theta\, b$ we have $a'\,\theta\, b'$. We assume $a \leqslant b$ and have $a \wedge b'\,\theta\, b \wedge a'$ and hence $b'\,\theta\,(b \wedge a') \vee b'$. As $b' \leqslant a'$ we get by the orthomodular $b' \vee (a' \wedge b) = a'$ hence $a'\,\theta\, b'$. If $a \nleqslant b$ then we consider $a \wedge b\,\theta\, a \vee b$ and derive $a' \wedge b'\,\theta\, a' \vee b'$.

REMARK. Every orthomodular lattice L is an algebra of a Malcev variety. If θ is a tolerance of L and is therefore compatible with the orthocomplementation of L then θ is also a congruence relation and vice versa [3], [9]. From [8, p. 663, Hilfssatz] we can derive

THEOREM 3. *Let L be a relatively complemented ortholattice. L is simple if and only if L has as a lattice only trivial tolerances.*

THEOREM 4. *If a modular lattice L of finite length has only trivial tolerances then L is atomistic.*

PROOF. For every element $a \in L$ we define $a^+ = \inf\{b | b \prec a\}$ if $a > 0$ and $a^+ = 0$ else. In a modular lattice of finite length we have $(a \vee b)^+ = a^+ \vee b^+$ [3, p. 269, Lemma 6.1(e)]. We consider the following binary relation $\rho = \{(a, b) | a \leqslant b, b^+ \leqslant a\}$ which is reflexive and compatible with join and meet. Obviously we have $D \subseteq \rho \subseteq R$ and as L has only trivial tolerance we conclude from Theorem 1 that $\rho = R$. As $(0, 1) \in R$ we have $1^+ = 0$ and hence L is coatomistic and complemented [1, Theorem IV.6].

Theorem 4 and the well-known results on modular geometric lattices give rise to the following theorems.

THEOREM 5. *Let L be a modular lattice of finite length. L is a projective geometry if and only if L has only trivial tolerances.*

THEOREM 6. *Let L be an arguesian lattice of finite length l, $l \geqslant 3$. L is isomorphic to the lattice of all subspaces of a vector space over some division ring if and only if L has only trivial tolerances.*

These results cannot be extended to lattices of infinite length. The restriction is necessary since the relation $a\,\rho\, b$ iff $a \leqslant b$ and $\operatorname{codim}_b(a) < \infty$ will generate a proper nontrivial tolerance relation on the subspace lattice of an infinite-dimensional projective space.

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