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ON THE ITERATED LOGARITHM LAW FOR LOCAL TIME

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ABSTRACT. If $s(t, x)$ is the local time of a Brownian motion, we show that $\theta(\alpha) = \limsup_{t\to\infty} \inf_{|x| \leqslant \alpha t^{1/2}(2\log\log t)^{-1/2}} s(t, x)(2t \log\log t)^{-1/2}$ satisfies

$$((1 - \alpha^{1/2}) \vee 0)^2 \leqslant \theta(\alpha) \leqslant (2\alpha)^{-1} \wedge 1.$$

In particular, it follows from a result of Kesten that

$$\limsup_{t\to\infty} s(t, x)(2t \log\log t)^{-1/2} = 1$$

for all x a.s.

1. Introduction. Suppose $B(t)$ is Brownian motion on a complete probability space $(\Omega, \mathfrak{F}, P)$ and $s(t, x) = (d/dx)\int_0^t I(B(s) \leqslant x)\, ds$ ($I(A)$ is the indicator function of A) is its jointly continuous local time. Since $s(t, 0)$ is identical in law to $\sup_{s\leqslant t} B(s)$, the law of the iterated logarithm implies that $\limsup_{t\to\infty} s(t, x)\phi(t)^{-1} = 1$ a.s. for each real x, where $\phi(t) = (2t|\log|\log t|\,|)^{1/2}$. In Kesten [1] it is shown that

$$\limsup_{t\to\infty} \sup_{x\in\mathbf{R}} s(t, x)\phi(t)^{-1} = 1 \quad \text{a.s.} \tag{1}$$

This implies that $\limsup_{t\to\infty} s(t, x)\phi(t)^{-1} \leqslant 1$ for all real x a.s. but leaves open the question as to whether or not there is equality for all x a.s. That there is equality for all x a.s. is an easy corollary of the following theorem.

THEOREM 1. *Let $\Psi(t) = t^{1/2}(2|\log|\log t|\,|)^{-1/2}$. There is a nonincreasing function $\theta(\alpha)$ $(\alpha \geqslant 0)$ such that*

(a) $\limsup_{t\to\infty} \inf_{|x|\leqslant\alpha\Psi(t)} s(t, x)\phi(t)^{-1} = \theta(\alpha)$ *a.s. for all* $\alpha \geqslant 0$,

(b) $\theta(\alpha) \leqslant (2\alpha)^{-1} \wedge 1$ *for all* $\alpha \geqslant 0$,

(c) $\theta(\alpha) \geqslant (1 - \alpha^{1/2})^2$ *for all* $\alpha \leqslant 1$. □

The method of proof is that in Kesten [1] but some simplification occurs due to the use of a maximal inequality for submartingales.

2. Main result.

NOTATION 2. If $a \geqslant 0$, let $T(a) = \inf\{t \geqslant 0 | s(t, 0) > a\}$.

LEMMA 3. *If $a \geqslant 0$, $s(T(a), x)$ is a martingale in $x \geqslant 0$ and satisfies*

$$E(e^{-\lambda s(T(a),x)}) = \exp\{-\lambda a(1 + 2\lambda|x|)^{-1}\} \qquad (\lambda \geqslant 0). \tag{2}$$

PROOF. By Knight [2], $s(T(a), x)$ is a diffusion in $x \geqslant 0$ with infinitesimal generator $2y d^2/dy^2$, and in particular is a nonnegative local martingale. Moreover,

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(2) is derived in the proof of Corollary 1.2 in Knight [2]. It follows from (2) that $E(s(T(a), x)) = a < \infty$ for all x and hence $s(T(a), x)$ is a supermartingale by Fatou's lemma. Since $E(s(T(a), x))$ is independent of x, we see that $s(T(a), x)$ must in fact be a martingale. □

PROOF OF THEOREM 1. Since $\limsup_{t\to\infty} \inf_{|x|\leqslant\alpha\Psi(t)} s(t, x)\phi(t)^{-1}$ is measurable with respect to the tail σ-field of a Brownian motion, a well-known zero-one law implies that the above expression is a.s. equal to a nonnegative constant $\theta(\alpha)$. Clearly $\theta(\alpha) \leqslant 1$ because $\limsup_{t\to\infty} s(t, 0)\phi(t)^{-1} = 1$ a.s. Moreover if $t > e$, then

$$\inf_{|x|\leqslant\alpha\Psi(t)} s(t, x)\phi(t)^{-1} \leqslant \int_{-\alpha\Psi(t)}^{\alpha\Psi(t)} s(t, x)\phi(t)^{-1}\, dx(2\alpha\Psi(t))^{-1}$$

$$\leqslant t(2\alpha\phi(t)\Psi(t))^{-1} = (2\alpha)^{-1}.$$

It remains to show (c). Fix $\alpha \in [0, 1)$ and $\alpha_1 \in (0, (1 - \alpha^{1/2})^2)$. Then choose $\alpha_2 \in (\alpha_1 \vee \alpha, 1)$ such that $\alpha_1 < (\alpha_2^{1/2} - \alpha^{1/2})^2$ or, equivalently, $\alpha < (\alpha_2^{1/2} - \alpha_1^{1/2})^2$. The usual proof of the law of the iterated logarithm allows us to choose $t > 1$ such that $P(T_k < t^k \text{ infinitely often}) = 1$, where $T_k = T(\alpha_2\phi(t^k))$. Therefore

$$P\Big(\inf_{|x|\leqslant\alpha\Psi(T_k)} s(T_k, x)\phi(T_k)^{-1} > \alpha_1 \text{ infinitely often}\Big)$$

$$\geqslant P\Big(\inf_{|x|\leqslant\alpha\Psi(t^k)} s(T_k, x) > \alpha_1\phi(t^k) \text{ and } T_k < t^k \text{ infinitely often}\Big)$$

$$\geqslant P\Big(\sup_{|x|\leqslant\alpha\Psi(t^k)} s(T_k, 0) - s(T_k, x) \geqslant (\alpha_2 - \alpha_1)\phi(t^k) \text{ only finitely often}\Big). \quad (3)$$

Use a maximal inequality for submartingales and Lemma 3 to see that if $\lambda \geqslant 0$, then

$$P\Big(\sup_{|x|\leqslant\alpha\Psi(t^k)} s(T_k, 0) - s(T_k, x) \geqslant (\alpha_2 - \alpha_1)\phi(t^k)\Big)$$

$$\leqslant 2P\Big(\sup_{0\leqslant x\leqslant\alpha\Psi(t^k)} s(T_k, 0) - s(T_k, x) \geqslant (\alpha_2 - \alpha_1)\phi(t^k)\Big)$$

$$\leqslant 2\exp\{-\lambda(\alpha_2 - \alpha_1)\phi(t^k)\}E\big(\exp\{\lambda(s(T_k, 0) - s(T_k, \alpha\Psi(t^k)))\}\big)$$

$$= 2\exp\{-\lambda(\alpha_2 - \alpha_1)\phi(t^k) + \lambda\alpha_2\phi(t^k) - \lambda\alpha_2\phi(t^k)(1 + 2\lambda\alpha\Psi(t^k))^{-1}\}$$

(by (2))

$$= 2\exp\{-\lambda\phi(t^k)(\alpha_2(1 + 2\lambda\alpha\Psi(t^k))^{-1} - \alpha_1)\}. \quad (4)$$

An elementary calculus argument shows that (4) has a minimum value of

$$2\exp\{-(\alpha_2^{1/2} - \alpha_1^{1/2})^2\alpha^{-1}|\log|\log t^k||\} \quad (5)$$

when $\lambda = ((\alpha_2\alpha_1^{-1})^{1/2} - 1)(2\alpha\Psi(t^k))^{-1}$. Since $(\alpha_2^{1/2} - \alpha_1^{1/2})^2 > \alpha$, (5) is summable over k and therefore

$$P\Big(\sup_{|x|\leqslant\alpha\Psi(t^k)} s(T_k, 0) - s(T_k, x) \geqslant (\alpha_2 - \alpha_1)\phi(t^k) \text{ only finitely often}\Big) = 1$$

by the Borel-Cantelli lemma. It follows from (3) that

$$\limsup_{k\to\infty} \inf_{|x|<\alpha\Psi(T_k)} s(T_k, x)\phi(T_k)^{-1} \geqslant \alpha_1 \quad \text{a.s.}$$

for all $\alpha_1 < (1 - \alpha^{1/2})^2$, and hence $\theta(\alpha) \geqslant (1 - \alpha^{1/2})^2$. □

Since $\theta(\infty) = 0$, if $h(t)$ satisfies $\lim_{t\to\infty} h(t)\Psi(t)^{-1} = +\infty$ then

$$\limsup_{t\to\infty} \inf_{|x|<h(t)} s(t, x)\phi(t)^{-1} = 0 \quad \text{a.s.},$$

and since $\theta(0^+) = 1$, if $h(t)$ satisfies $\lim_{t\to\infty} h(t)\psi(t)^{-1} = 0$ then

$$\limsup_{t\to\infty} \inf_{|x|<h(t)} s(t, x)\phi(t)^{-1} = 1 \quad \text{a.s.}$$

This latter result (with $\lim_{t\to\infty} h(t) = \infty$), coupled with (1), gives us the following corollary.

COROLLARY 4. *For* ω *outside a single null set*, $\limsup_{t\to\infty} s(t, x)\phi(t)^{-1} = 1$ *for all* x. □

REMARK 5. A trivial modification of the proof of Theorem 1 shows that for all $\alpha \geqslant 0$ there is a constant $\theta^1(\alpha)$ satisfying (b) and (c) of Theorem 1 and also

$$\limsup_{t\to 0^+} \inf_{|x|<\alpha t^{1/2}(2\log\log t^{-1})^{-1/2}} s(t, x)(2t \log\log(t^{-1}))^{-1/2} = \theta^1(\alpha) \quad \text{a.s.} \quad \square$$

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