

EXTENDING THE CONCEPT OF GENUS TO DIMENSION N

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ABSTRACT. Some graph-theoretical tools are used to introduce the concept of "regular genus" $\mathcal{G}(M_n)$, for every closed n -dimensional PL-manifold M_n . Then it is proved that the regular genus of every surface equals its genus, and that the regular genus of every 3-manifold M_3 equals its Heegaard genus, if M_3 is orientable, and twice its Heegaard genus, if M_3 is nonorientable. A geometric approach, and some applications in dimension four are exhibited.

1. Definitions and notations. Let $\Gamma = (V, E)$ be a regular multigraph of degree $n + 1$, $\gamma: E \rightarrow \Delta_n = \{i \in \mathbb{Z} | 0 \leq i \leq n\}$ an $(n + 1)$ -line-colouring of Γ [Ha, p. 133]. Such a pair (Γ, γ) is said to be an $(n + 1)$ -coloured graph. For every subset \mathcal{B} of Δ_n , $\Gamma_{\mathcal{B}}$ will denote the subgraph $(V, \gamma^{-1}(\mathcal{B}))$; further, for every colour $c \in \Delta_n$, \hat{c} will denote the set $\Delta_n - \{c\}$.

A 2-cell imbedding $\mathbf{i}: |\Gamma| \rightarrow F$ [W, p. 40] of an $(n + 1)$ -coloured graph (Γ, γ) on a closed surface F is called a *regular imbedding* if its regions are bounded by 2-coloured cycles. Moreover, \mathbf{i} is called a *strongly-regular imbedding* if there exists a cyclic permutation $\varepsilon = (\varepsilon_0, \dots, \varepsilon_n)$ of Δ_n , so that each region is bounded by a component of one of the subgraphs $\Gamma_{\{\varepsilon_i, \varepsilon_{i+1}\}}$, i being an integer modulo $n + 1$.

It is easy to see that every strongly-regular imbedding is regular. Conversely it is proved in [G₄] that every regular imbedding of a 3- or 4-coloured graph is strongly regular. This is not true in general, as it is easy to check.

In the present work, we only consider strongly-regular imbeddings, as they seem to apply better to the geometric situation we wish to represent.

For the sake of conciseness, we shall always omit the word "strongly". So we shall simply call *regular* the *strongly-regular* imbeddings of [G₄].

A pseudocomplex [HW, p. 49] $K = K(\Gamma)$ of dimension n can be associated to every $(n + 1)$ -coloured graph, so that Γ becomes its dual 1-skeleton; for the construction, compare [G₂]. If, for every colour $c \in \Delta_n$, the subgraph $\Gamma_{\hat{c}}$ is connected, then $K(\Gamma)$ has exactly $n + 1$ vertices, and both K and Γ are said to be *contracted*.

Now let M be a closed n -manifold, (Γ, γ) a contracted $(n + 1)$ -coloured graph, such that the space $|K(\Gamma)|$ of its associated complex is homeomorphic with M ; then

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(Γ, γ) is called a *crystallisation* and K a *contracted triangulation* of M . A theorem of Pezzana [P₁], [P₂] assures that every closed n -manifold admits a contracted triangulation, and therefore it can be represented by a crystallisation. A faster construction for the case of a 2-fold branched covering of S^3 is shown in [F].

By a *Heegaard splitting of genus n* for a closed 3-manifold M is meant a pair $(\mathcal{Q}, \mathcal{Q}')$ of handlebodies of genus n [He, p. 15], such that $\mathcal{Q} \cap \mathcal{Q}' = \partial\mathcal{Q} = \partial\mathcal{Q}'$, and $\mathcal{Q} \cup \mathcal{Q}' = M$. Their common boundary $F = \partial\mathcal{Q} = \partial\mathcal{Q}'$ is called the *Heegaard surface* of the splitting. Compare also [ST] and [M].

All maps and spaces considered in this work are in the polyhedral category; we refer to [Gl] for the main tools about PL-topology and to [Ha] for the used graph-theoretical arguments.

2. The regular genus of a closed n -manifold. Let (Γ, γ) be a connected $(n + 1)$ -coloured graph. It is shown in [G₄] that Γ admits exactly $n!/2$ regular imbeddings in some orientable (resp. nonorientable) surface, iff Γ is a bipartite (resp. nonbipartite) graph. Besides, no regular nonorientable (resp. orientable) imbedding may exist for a bipartite (resp. nonbipartite) $(n + 1)$ -coloured graph. So the following definition is well posed.

DEFINITION 1. By the *regular genus* $\rho(\Gamma)$ of (Γ, γ) we mean the smallest integer n , such that Γ regularly imbeds on the (orientable or nonorientable) closed surface of genus n .

As bipartite (resp. nonbipartite) graphs admit only orientable (resp. nonorientable) such imbeddings, then $\rho(\Gamma)$ will be *orientable* or *nonorientable* according to Γ being a bipartite or a nonbipartite graph.

Now let M be a closed n -manifold, and $\mathfrak{C} = \mathfrak{C}(M)$ the class of all the crystallisations of M . Observe that \mathfrak{C} is nonempty by Pezzana's theorem. Further, any two crystallisations of \mathfrak{C} are joined by a finite sequence of moves, described in [FG].

DEFINITION 2. With the prior notations, set

$$\mathfrak{G}(M) = \min\{\rho(\Gamma) \mid (\Gamma, \gamma) \in \mathfrak{C}(M)\}.$$

We shall call $\mathfrak{G}(M)$ the *regular genus* of the n -manifold M .²

3. The main results.

THEOREM. (A) Let M_2 be a closed surface, and $g(M_2)$ its genus. Then $\mathfrak{G}(M_2) = g(M_2)$.

(B) Let M_3 be a closed 3-manifold, and $\mathfrak{H}(M_3)$ its Heegaard genus.³ Then

- (i) $\mathfrak{G}(M_3) = \mathfrak{H}(M_3)$ if M_3 is orientable;
- (ii) $\mathfrak{G}(M_3) = 2\mathfrak{H}(M_3)$ if M_3 is nonorientable.

²A similar concept can be defined for every space representable by $(n + 1)$ -coloured graphs.

³The Heegaard genus of a 3-manifold M_3 is defined as the smallest integer h , such that M_3 admits a Heegaard splitting of genus h . Observe that the Heegaard surface of the splitting has genus h , if orientable, or $2h$, if nonorientable.

PROOF OF THEOREM A. Every 3-coloured graph (in particular every surface-crystallisation) represents a closed surface M_2 if and only if it regularly imbeds in M_2 [G_4 , Corollary 4]. Moreover, this imbedding is unique, up to equivalences [G_4 , Proposition 3]. Thus $\mathcal{G}(M_2) = g(M_2)$, for every closed surface M_2 . \square

The proof of Theorem (B) needs some notations and two lemmas.

Let M_3 be a closed 3-manifold and $(\mathcal{Q}, \mathcal{Q}')$ a Heegaard splitting of genus n for M_3 . By a *Heegaard diagram* arising from $(\mathcal{Q}, \mathcal{Q}')$ is meant a triple $(F, \mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n))$, where $F = \partial\mathcal{Q} = \partial\mathcal{Q}'$ is the Heegaard surface of the splitting, and \mathbf{x}, \mathbf{y} are two disjoint systems of curves on F , satisfying the following properties.

- (a) each curve x_i (resp. y_i) bounds a 2-ball X_i of \mathcal{Q} (resp. Y_i of \mathcal{Q}').
- (b) \mathcal{Q} (resp. \mathcal{Q}'), cut open along the 2-balls X_i (resp. Y_i), $i = 1, \dots, n$, is a 3-ball.

Now let (Γ, γ) be a crystallisation of M_3 . As shown in [G_4 , Proposition 11], for every pair of colours $\alpha, \beta \in \Delta_3$, Γ admits a regular imbedding $i_{\alpha\beta}: |\Gamma| \rightarrow F_{\alpha\beta}$, where $F_{\alpha\beta}$ is a Heegaard surface of M_3 , determining the splitting $(\mathcal{Q}_{\alpha\beta}, \mathcal{Q}'_{\alpha\beta})$. Also let ξ_0, \dots, ξ_n be the components of $\Gamma_{\{\alpha, \beta\}}$, ξ'_0, \dots, ξ'_n the ones of $\Gamma_{\{\alpha', \beta'\}}$, α', β' being the remaining two colours.

Set $x_i = i_{\alpha\beta}(\xi_i)$ and $y_i = i_{\alpha\beta}(\xi'_i)$, for $i \in \Delta_n$; finally let

$$\mathbf{x}(\hat{j}) = \{x_0, \dots, \hat{x}_j, \dots, x_n\}, \quad \mathbf{y}(\hat{k}) = \{y_0, \dots, \hat{y}_k, \dots, y_n\}.$$

LEMMA 3. For every $j, k \in \Delta_n$, the triple $(F_{\alpha\beta}, \mathbf{x}(\hat{j}), \mathbf{y}(\hat{k}))$ is a Heegaard diagram of M_3 , arising from the splitting $(\mathcal{Q}_{\alpha\beta}, \mathcal{Q}'_{\alpha\beta})$.

PROOF. It is shown in [P_2], and subsequently in [G_4] that $(\mathcal{Q}_{\alpha\beta}, \mathcal{Q}'_{\alpha\beta})$ has genus n , $n+1$ being the number of components of $\Gamma_{\{\alpha, \beta\}}$. Further, each curve x_i (resp. y_i) bounds a 2-ball X_i of $\mathcal{Q}_{\alpha\beta}$ (resp. Y_i of $\mathcal{Q}'_{\alpha\beta}$), and $\mathcal{Q}_{\alpha\beta}$ (resp. $\mathcal{Q}'_{\alpha\beta}$), cut open along all the 2-balls X_i (resp. Y_i), splits into the disjoint union of two 3-balls. It follows that $\mathcal{Q}_{\alpha\beta}$ (resp. $\mathcal{Q}'_{\alpha\beta}$), cut open along all the X_i (resp. Y_i), but one, arbitrarily chosen, is a 3-ball. The proof is therefore complete. \square

LEMMA 4. Let $(\mathcal{Q}, \mathcal{Q}')$ be a Heegaard splitting of M_3 , $F = \partial\mathcal{Q} = \partial\mathcal{Q}'$ its Heegaard surface. Then there exists a crystallisation (Γ, γ) of M_3 which regularly imbeds in F .

PROOF. Let $(F, \mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n))$ be a Heegaard diagram, arising from $(\mathcal{Q}, \mathcal{Q}')$, n being the genus of the splitting. It becomes useful, in the construction of (Γ, γ) , to take the usual representation of the diagram in the ordinary euclidean space (see [S]).

For, let S^2 be the 2-sphere, represented as the (x, y) -plane, plus a point at infinity. For every integer r , let $P_r = (r, -1)$, $P'_r = (r, 1)$, $\mathcal{X}_r, \mathcal{X}'_r$ the balls of radius $1/4$ and centers at P_r and P'_r respectively, $\mathring{\mathcal{X}}_r, \mathring{\mathcal{X}}'_r$ their interiors, $\mathfrak{x}_r, \mathfrak{x}'_r$ their boundaries. By Λ_{2n} is meant the surface $S^2 - \bigcup_{r=1}^n \mathring{\mathcal{X}}_r - \bigcup_{r=1}^n \mathring{\mathcal{X}}'_r$. Now let $f: \Lambda_{2n} \rightarrow F$ be a map, one-to-one everywhere, except that each point of $x_r, r \in \Delta_n - \{0\}$, has two points, one of \mathfrak{x}_r and one of \mathfrak{x}'_r , as inverse image.

Also let Λ_+ (resp. Λ_-) be the subset of Λ_{2n} consisting of all the points with nonnegative (resp. nonpositive) ordinate, plus the point at infinity.

By isotoping, if necessary, the y -curves, we can suppose that the following three conditions hold.

(a) For every $i \in \Delta_n - \{0\}$, $f^{-1}(y_i)$ is the disjoint union of a finite set of arcs $\{\alpha_i^j\}$, each meeting a circle only at its endpoints.

(b) Let α_i^j be anyone of the previous arcs, and x_0 the x -axis plus the point at infinity; then $\alpha_i^j \cap \Lambda_+$ (resp. $\alpha_i^j \cap \Lambda_-$), if nonempty, is the disjoint union of a finite set of arcs $\{b_{ih}^j\}$ (resp. $\{c_{ik}^j\}$), none of which meets x_0 at an inner point.

Each circle x_r (resp. x'_r), $r \in \Delta_n - \{0\}$, is split by the endpoints of the arcs α_i^j into the union of a finite set of arcs $\{d_{rs}\}$ (resp. $\{d'_{rs}\}$). Let \mathcal{E}' be the union of the families of arcs $\{\alpha_i^j\}$, $\{d_{rs}\}$, $\{d'_{rs}\}$, and \mathcal{V}' the set of their endpoints.

By condition (a), we can consider the pseudograph $\Gamma' = (V', E')$ (multiple edges and loops may occur), where the vertex-set V' is in bijection with \mathcal{V}' , and two vertices v, w are joined by an edge of E' iff the corresponding elements of \mathcal{V}' are the endpoints of the same arc of \mathcal{E}' .

With a slight abuse of notation, we can suppose Γ' imbedded in Λ_{2n} , by identifying (V', E') with $(\mathcal{V}', \mathcal{E}')$. With the prior convention, we can state the third condition:

(c) None of the connected components of Γ' lies entirely in Λ_+ or Λ_- .

Let $\{d_{0m}\}$ be the arcs into which the circle x_0 is subdivided by the endpoints of the arcs $\{b_{ih}^j\}$ and $\{c_{ik}^j\}$. By conditions (b), (c), we can consider a new pseudograph $\Gamma'' = (\mathcal{V}'', \mathcal{E}'')$, where \mathcal{E}'' is the union of the families of arcs $\{b_{ih}^j\}$, $\{c_{ik}^j\}$, $\{d_{rs}\}$, $\{d'_{rs}\}$, and \mathcal{V}'' the set of their endpoints.

Set $\bar{\mathcal{V}} = f(\mathcal{V}'')$, $\bar{\mathcal{E}} = f(\mathcal{E}'')$. Again a pseudograph $\bar{\Gamma} = (\bar{V}, \bar{E})$, with (\bar{V}, \bar{E}) identified with $(\bar{\mathcal{V}}, \bar{\mathcal{E}})$, can be considered.

Observe that the result of the previous construction is the adding of a new cycle $x_0 = f(x_0)$ to the family x , and the rearrangement of the y -curves, in order to get $\bar{\Gamma}$, imbedded in F , and whose inverse image under f satisfies properties (a)–(c).

Now let $Q_r = (-1, r)$, $Q'_r = (1, r)$, $\mathfrak{y}_r, \mathfrak{y}'_r, \mathfrak{y}''_r, \mathfrak{y}'''_r, \eta_r, \eta'_r, \Lambda'_{2n}, f': \Lambda'_{2n} \rightarrow F, \Lambda'_+, \Lambda'_-$ defined as below, simply by exchanging the x -axis with the y -axis. A perfectly analogous construction can be done, starting from $\bar{\Gamma}$, changing the rôle of the sets x and y . The result is a multigraph $\Gamma''' = (\mathcal{V}''', \mathcal{E}''')$, imbedded in Λ'_{2n} , and a multigraph $\Gamma = (V, E)$, regular of degree four, with (V, E) identified with $(f'(\mathcal{V}'''), f'(\mathcal{E}'''))$.

Observe that Γ can be obtained from $\bar{\Gamma}$ simply by adding a suitable cycle y_0 to the family of cycles, whose images onto F are the y -curves.

In order to define the colouring $\gamma: E \rightarrow \Delta_3$, it suffices to consider the two representations of Γ on Λ'_{2n} and Λ_{2n} , by means of the maps f', f respectively. The first is $\Gamma''' = (\mathcal{V}''', \mathcal{E}''')$ while the second, Γ'''' say, is obtained from Γ''' by adding the inverse image under f of the cycle y_0 .

Now, label by “0” (resp. by “2”) all the edges of Γ''' , lying in Λ'_+ (resp. Λ'_-) but not belonging to the cycles $\eta_0, \eta_r, \eta'_r, r \in \Delta_n - \{0\}$. Analogously, label by “1”

(resp. by “3”) all the edges of Γ''' , lying in Λ_+ (resp. Λ_-), but not belonging to the cycles $r_0, r_i, r'_i, i \in \Delta_n - \{0\}$.

The pair (Γ, γ) , where γ is the induced colouring on the edges of Γ , is a 4-coloured graph, whose realisation on F provides a regular inbedding $i: |\Gamma| \rightarrow F$.⁴ Moreover, condition (c) proves that (Γ, γ) is a *contracted* graph.

By the construction itself, (Γ, γ) turns out to be a 3-manifold crystallisation, as each of the four subgraphs $\Gamma_c, c \in \Delta_3$, can be regularly imbedded in $\Lambda_+, \Lambda_-, \Lambda'_+, \Lambda'_-$, and therefore $\rho(\Gamma_c) = 0$, for every $c \in \Delta_3$ (compare [P₃, Proposition 10] or [G₄, Corollary 16-e’]).

We have only to show that (Γ, γ) is a crystallisation of M_3 . For, let $\{\xi_0, \dots, \xi_n\}$ (resp. $\{\xi'_0, \dots, \xi'_n\}$) be the set of connected components of the subgraph $\Gamma_{\{0,2\}}$ (resp. $\Gamma_{\{1,3\}}$), whose image on F is $\tilde{x} = x \cup \{x_0\}$ (resp. $\tilde{y} = y \cup \{y_0\}$).

Thus, $(F, \tilde{x}(\hat{0}), \tilde{y}(\hat{0}))$ is (equivalent to) the original Heegaard diagram (F, x, y) of M_3 . The proof of Lemma 4 is thereby complete. \square

Figures 1(a), 1(b) and 1(c) illustrate the construction of Lemma 4 applied to the classical Heegaard diagram of the Poincaré space (compare [Po]), which is also reproduced in [R, p. 245].

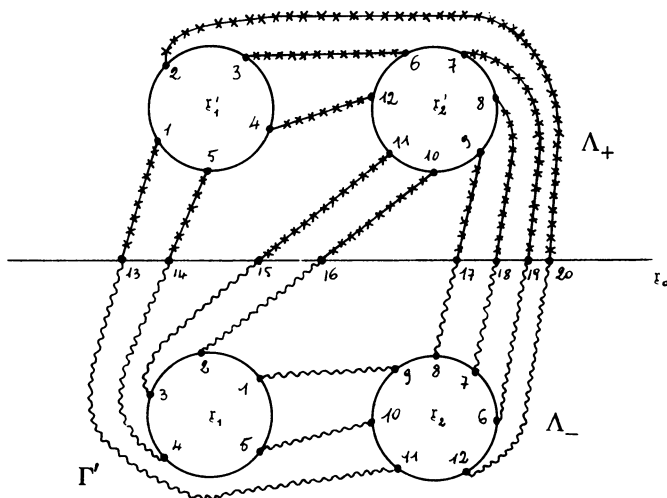


FIGURE 1(a)

PROOF OF THEOREM B. This is an easy consequence of the previous two lemmas.

\square

REMARK. Lemmas 3 and 4 give an algorithm to get a crystallisation from a Heegaard diagram of a closed 3-manifold, and conversely.

⁴In the terminology of [G₄], i is induced by the fundamental cyclic permutation of Δ_3 .

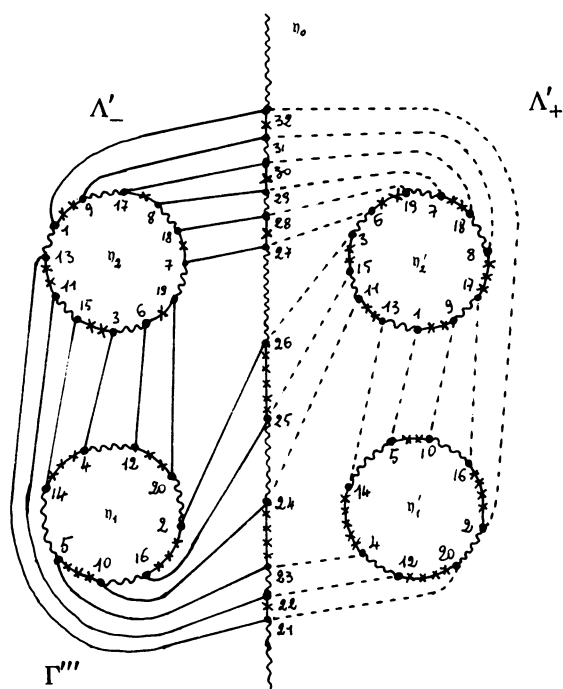


FIGURE 1(b)

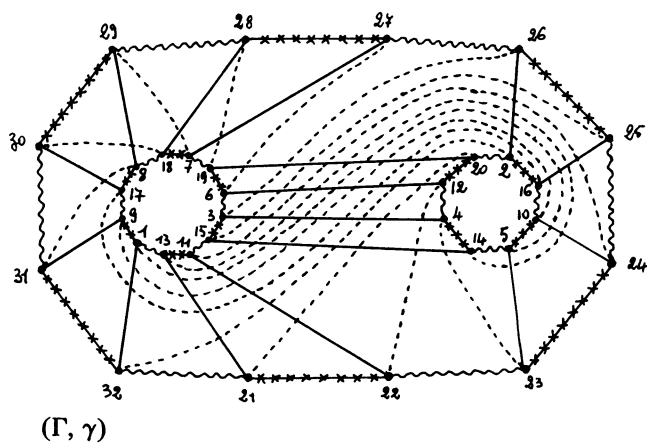


FIGURE 1(c)

4. A geometric approach. Let M_n be a closed PL n -manifold, K a contracted triangulation of M_n , v_0, \dots, v_n the $n+1$ vertices of K , $\text{Sd } K$ its first barycentric subdivision. If A^n is an n -simplex of K , i, j two different elements of Δ_n , let us call $A^{n-2}(\hat{i}, \hat{j})$ the $(n-2)$ -face of A^n not containing the two vertices v_i and v_j .

Now, for every cyclic permutation $\varepsilon = (\varepsilon_0, \dots, \varepsilon_n)$ of Δ_n , define H_ε to be the subcomplex of $\text{Sd } K$, obtained as follows:

(a) for every n -simplex A^n , and for every $i \in \mathbb{Z}_{n+1}$,⁵ consider the barycenter $\underline{a}(\hat{e}_i, \hat{e}_{i+1})$ of $A^{n-2}(\hat{e}_i, \hat{e}_{i+1})$; then call $\underline{a}(\hat{e}_i)$ the barycenter of the $(n-1)$ -face opposite to the vertex v_e ;

(b) consider the 1-sphere Σ , consisting of all the 1-simplexes of $\text{Sd } A^n$, joining $\underline{a}(\hat{e}_i)$ with either $\underline{a}(\hat{e}_i, \hat{e}_{i+1})$ or $\underline{a}(\hat{e}_{i-1}, \hat{e}_i)$, i being an integer modulo $n+1$;

(c) set $\mathcal{O} = \underline{a} * \Sigma$, \underline{a} being the barycenter of A^n , and define H_e to be the union of all the 2-balls \mathcal{O} 's so obtained.

It is easy to see that the space $F_e = |H_e|$ is a closed surface, PL-imbedded in M_n . We shall call each F_e a *regular surface* of M_n associated to K and ϵ .

Observe that, for $n = 3$, each regular surface is actually a Heegaard surface, and conversely.

Now set $\tilde{\mathcal{G}}(M_n) = \min\{g(F) | F \text{ is a regular surface of } M_n\}$ where $g(F)$ denotes, as before, the genus of the surface F .

PROPOSITION 5. *With the previous notations, we have $\tilde{\mathcal{G}}(M_n) = \mathcal{G}(M_n)$.*

PROOF. Suppose $F = F_e(K)$, a regular surface of M_n associated to the contracted complex K and the cyclic permutation ϵ . An easy calculation shows that the Euler characteristic of F is

$$\chi(F) = \sum_{i \in \mathbb{Z}_{n+1}} q^{n-2}(\hat{e}_i, \hat{e}_{i+1}) + (1-n)q^n/2$$

where q^n is the number of n -simplexes of K , and $q^{n-2}(\hat{e}_i, \hat{e}_{i+1})$ denote the number of $(n-2)$ -simplexes of K which do not contain vertices v_e and v_s .

The desired result follows immediately from the definition of regular genus of M_n , and the formula of [G₄, Propositions 19, 23]. \square

REMARK. A direct geometric proof of Proposition 5 can be accomplished by showing that (i) any crystallisation (Γ, γ) of M_n regularly imbeds into all the regular surfaces associated to $K(\Gamma)$, (ii) if (Γ, γ) regularly imbeds in a surface F , then F can be imbedded in M_n , so that it becomes a regular surface of it.

Such a construction, for dimension 3, is exposed in [G₄, Proposition 12].

5. Applications to dimension four. Let M_n be a closed orientable (resp. nonorientable) n -dimensional manifold; for any contracted triangulation K of M_n , we shall describe a family of "handlebody decompositions" for M_n . For every pair of vertices v_i, v_j of K , $i, j \in \Delta_n$, consider the contracted subcomplexes K_{ij} and K_{ij}^{\sim} , generated by the vertex-sets $\{v_i, v_j\}$ and $\{v_k | k \in \Delta_n - \{i, j\}\}$ respectively.

Also let H_{ij} be the largest subcomplex of $\text{Sd } K$, disjoint from $\text{Sd } K_{ij} \cup \text{Sd } K_{ij}^{\sim}$. H_{ij} is a closed orientable (resp. nonorientable, if $n > 2$) $(n-1)$ -manifold, which splits $\text{Sd } K$ into two complementary subcomplexes, L_{ij} and L_{ij}^{\sim} say, having H_{ij} as common boundary; moreover, they are regular neighbourhoods of $\text{Sd } K_{ij}$ and $\text{Sd } K_{ij}^{\sim}$ respectively, and $|L_{ij}|$ is an n -dimensional handlebody (i.e. an n -ball with 1-handles attached upon).

⁵ \mathbb{Z}_h denotes the cyclic group of order h .

Suppose now $n = 2k + 1$ (resp. $n = 2k$), and consider any partition $\{\alpha_0, \alpha_1\}, \{\alpha_2, \alpha_3\}, \dots, \{\alpha_{2k}, \alpha_{2k+1}\}$ (resp. $\{\alpha_0, \alpha_1\}, \{\alpha_2, \alpha_3\}, \dots, \{\alpha_{2k}\}$) of Δ_n ; then call \mathcal{Q}_i the space $|L_{2\alpha_i, 2\alpha_i+1}|$ for $i \in \Delta_{k-1}$, and \mathcal{Q}_k the space $|L_{2\alpha_k, 2\alpha_k+1}|$ or $|L_{2\alpha_k}|$, according with n being an odd or an even number.

The $(k + 1)$ -tuple $(\mathcal{Q}_0, \mathcal{Q}_1, \dots, \mathcal{Q}_k)$ of handlebodies satisfies the following properties.

- (a) $\bigcup_{r \in \Delta_k} \mathcal{Q}_r = M_n$,
- (b) for every $j \in \Delta_k$, $\mathcal{Q}_j \cap (\bigcup_{i \neq j} \mathcal{Q}_i) = \partial \mathcal{Q}_j$,
- (c) if $n = 2k$, then \mathcal{Q}_k is an n -ball.

It is easy to see that the $(k + 1)$ -tuple mentioned previously reduces to a Heegaard splitting for dimension three. For this reason, $(\mathcal{Q}_0, \dots, \mathcal{Q}_k)$ will be called a *generalized Heegaard decomposition* of M_n . More details are available in [G₁].

The following proposition binds the concepts of "regular surface" and of "generalized Heegaard decomposition", in dimension four.

PROPOSITION 6. *Suppose M_4 a closed 4-dimensional manifold. Then, for every regular surface F of M_4 , there exists a generalized Heegaard decomposition $(\mathcal{Q}_0, \mathcal{Q}_1, \mathcal{Q}_2)$ of M_4 , such that F is a Heegaard surface of \mathcal{Q}_0 .*

COROLLARY 7. *Let M_4 be a closed orientable (resp. nonorientable) 4-manifold, and let $\mathfrak{G}(M_4) = h$, with $h \geq 0$ (resp. $\mathfrak{G}(M_4) = 2h$, with $h > 0$). Then $\pi_1(M_4)$ is a quotient of the free group on h generators. In particular,*

- (a) *if $\mathfrak{G}(M_4) = 0$, then M_4 is simply connected,*
- (b) *if $\mathfrak{G}(M_4) = 1$ and M_4 is orientable (resp. $\mathfrak{G}(M_4) = 2$ and M_4 is nonorientable), then $\pi_1(M_4)$ is a cyclic group.*

PROOF. Let (Γ, γ) be a crystallisation of M_4 , ϵ a cyclic permutation of Δ_4 , and F the regular surface of M_4 , associated with $K(\Gamma)$ and ϵ . It follows from Proposition 5 that there exists a regular imbedding $i: |\Gamma| \rightarrow F$. As shown in [G₄], i is induced by a cyclic permutation of Δ_4 , which is exactly ϵ , as it is not hard to check. We can suppose, w.l.o.g., ϵ induced by the fundamental permutation $(0, 1, 2, 3, 4)$.

Now, let us consider the generalized Heegaard decomposition $(\mathcal{Q}_0, \mathcal{Q}_1, \mathcal{Q}_2)$ of M_4 , relative to the partition $\{0, 2\}, \{1, 3\}, \{4\}$ of Δ_4 ; also call \mathcal{Q} the 3-dimensional ball-complex, whose 3-balls are the intersections of $\partial \mathcal{Q}_0$ with each 4-simplex of K , and \mathcal{Q}' the 2-dimensional ball-complex, whose 2-balls are the (pentagonal) regions of the dual graph of the imbedding i in F . $|\mathcal{Q}|$ and $|\mathcal{Q}'|$ are obviously homeomorphic to $\partial \mathcal{Q}_0$ and F respectively.

Now, each 2-ball of \mathcal{Q}' is imbedded in a 3-ball of \mathcal{Q} , as indicated in Figure 7 of [G₄]; the induced imbedding, $F \hookrightarrow \partial \mathcal{Q}_0$, makes F a Heegaard surface of $\partial \mathcal{Q}_0$. This completes the proof of Proposition 6.

To prove Corollary 7, we first remark that \mathcal{Q}_0 is a 4-dimensional handlebody of genus λ (i.e. a 4-ball with λ 1-handles attached upon), where $\lambda + 1$ is the number of 1-simplexes of K_{02} , or equivalently, the number of connected components of the subgraph $\Gamma_{\{1,3,4\}}$. Its boundary $\partial \mathcal{Q}_0$ is thereby PL-homeomorphic either to

$\lambda \# S^1 \times S^2$, if orientable, or to $\lambda \# S^1 \times S^2$, if nonorientable.⁶ By the prior proposition, if $g(F) = \mathcal{G}(M_4)$, then $\lambda \leq h$ in both the orientable and the nonorientable cases. Now, the main result of [G₃] states that $\pi_1(M_4)$ admits a presentation with λ generators, and the proof is therefore complete. \square

As an easy consequence of the previous construction, we can state the following.

COROLLARY 8. *If M_4 is a closed nonorientable 4-manifold, then $\mathcal{G}(M_4)$ is an even positive number.* \square

We conjecture that Proposition 6 and Corollaries 7 and 8 can be generalized to all dimensions.

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⁶ $\lambda \# S^1 \times S^2$ (resp. $\lambda \# S^1 \times S^2$) denotes the connected sum of λ copies of $S^1 \times S^2$ (resp. of the nonorientable 2-sphere bundle over S^1).