

APPROXIMATION OF CONTINUOUS FUNCTIONS ON PSEUDOCOMPACT SPACES

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ABSTRACT. If \mathcal{S}^* is the family of subrings of $C^*(X)$ such that if $S \in \mathcal{S}^*$, S contains the constant functions and is closed under uniform convergence, then the following are equivalent for a space (X, \mathcal{T}) .

- (a) (X, \mathcal{T}) is pseudocompact.
- (b) If $S \in \mathcal{S}^*$ functionally separates points and zero sets, S generates (X, \mathcal{T}) .
- (c) If $S \in \mathcal{S}^*$ functionally separates zero sets, $S = C^*(X)$.
- (d) If $S \in \mathcal{S}^*$ generates the zero sets on (X, \mathcal{T}) , $S = C^*(X)$.
- (e) If $f \in S \in \mathcal{S}^*$ and $Z(f) = \emptyset$ then $1/f \in S$ (even when it is required that S generate the topology).
- (f) If $f \in S \in \mathcal{S}$ then $|f| \in S$ (even when it is required that S generate the topology).

1. Introduction. Since Weierstrass [14] proved his classic theorem in 1885 that any continuous function on a closed interval can be approximated uniformly by polynomials, there have been many generalizations of this theorem. Stone [13, 1937] replaced closed intervals by arbitrary compact T_2 spaces and replaced polynomials by rings of continuous functions to the real line that included the constant functions and functions that distinguished points. Hewitt [8, 1947] showed that for Tychonoff spaces you could not weaken compactness in Stone's theorem: however, he [8] showed that if the ring completely separated points and zero sets (disjoint zero sets) compactness may be replaced by almost compactness (compactness may be replaced by Tychonoff). Thus if we let $S^*(\mathcal{S})$ consist of all subrings of $C^*(X)$ ($C(X)$) that include constant functions and are closed under uniform convergence, we have

THEOREM A (HEWITT). *The following are equivalent for a Tychonoff space X .*

- (a) X is almost compact.
- (b) For $S \in \mathcal{S}^*$ such that S completely separates points and zero sets, $S = C^*(X)$.

(For a definition of almost compact, see [5, p. 95].)

We may replace completely separates by functionally separates in Theorem A. (See Theorem 1, §2.)

DEFINITION 1. A ring S of continuous functions to the reals functionally separates two sets (necessarily disjoint) A and B if there is $f \in S$ such that $f(A) \cap f(B) = \emptyset$. Completely separates would require that $\overline{f(A)} \cap \overline{f(B)} = \emptyset$. See

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[5, p. 16]. A ring S generates the zero sets of X if for $Z \in \mathcal{Z}(X)$ there exists $f \in S$ such that $Z = f^{-1}(0)$.

THEOREM B. (HEWITT). *If $S \in \mathfrak{S}^*$ for a Tychonoff space X is such that S completely separates zero sets then $S = C^*(X)$.*

In 1964, Mrowka showed that if $S \in \mathfrak{S}$ completely separates zero sets and, $Z(f) = \emptyset$, then $S = C(X)$. Hager and Johnson [7] then proved the following.

THEOREM C. *The following are equivalent for a Tychonoff space X .*

- (a) *X is Lindelöf or almost compact.*
- (b) *If $S \in \mathfrak{S}$ is such that $1/f \in S$ whenever $f \in S$ and $Z(f) = \emptyset$ and S completely separates points and zero sets then $S = C(X)$.*

Banaschewski [2, 1957] and Stephenson [12, 1968] studied the Stone-Weierstrass Theorem in more general spaces than Tychonoff spaces.

THEOREM D. *For a functionally Hausdorff space (see Definition 2 in §3) the following are equivalent.*

- (a) *The weak topology for X is compact.*
- (b) *For $S \in \mathfrak{S}$ such that S completely separates points, $S = C^*(X)$.*

Our point of departure will be the concept of functional separation and its effect on Stone-Weierstrass type theorems in pseudocompact spaces. The following lemma of Isiwata [9] will be useful.

LEMMA A. *If Z is a zero set of a pseudocompact space X and f is continuous on X and $f(Z) > 0$ then there exists $\epsilon > 0$ such that $f(z) > \epsilon > 0$ for $z \in Z$ (i.e. $f(Z) \succ \epsilon > 0$).*

A subset satisfying the above property with respect to $C^*(X)$ will be said to satisfy the Isiwata property.

LEMMA B. *Let f be continuous on X and let $Z \subset X$ such that Z satisfies the Isiwata property with respect to $C^*(X)$. Then $f(Z)$ satisfies the Isiwata property with respect to $C^*(f(X))$.*

PROOF. Suppose $g \in C^*(f(X))$ and $g(f(Z)) > 0$. Since $gf > 0$ on Z , $gf(Z) \succ \epsilon > 0$, so $g(f(Z)) \succ \epsilon > 0$.

REMARKS. In general the terminology is from Gillman and Jerison [5] and most of the historical remarks are in Willard [15].

2. The role of pseudocompactness.

LEMMA 1. *The following are equivalent for a Tychonoff space X .*

- (a) *X is pseudocompact.*
- (b) *Every z -embedding of X in a Tychonoff space is a C^* -embedding.*
- (c) *Every z -embedding of X in a compact T_2 -space is a C^* -embedding.*
- (d) *Every z -embedding of X in a Tychonoff one point extension is a C^* -embedding.*

PROOF. (a) \rightarrow (b) is due to Blair and Hager [3]. (b) \rightarrow (c) and (b) \rightarrow (d) are immediate. (d) \rightarrow (a). If X is not pseudocompact there is a zero set Z of βX contained in $\beta X \sim X$. Then X is z -embedded in $X \cup [z]$, the quotient space obtained from $X \cup Z$ by identifying Z with a point, but X is clearly not C^* -embedded in $X \cup [z]$. The z -embedding of X in $X \cup [z]$ follows from the fact that if H is a zero set of X then $H \cup Z$ is a zero set of $X \cup Z$ and $H \cup [z]$ will then be a zero set of $X \cup [z]$. (c) \rightarrow (a) is similar to (d) \rightarrow (a).

THEOREM 1. *The following are equivalent for a Tychonoff space (X, \mathfrak{T}) .*

- (a) (X, \mathfrak{T}) is pseudocompact.
- (b) If $S \in \mathfrak{S}^*$ is such that S functionally separates points and zero sets then S generates (X, \mathfrak{T}) .
- (c) If $S \in \mathfrak{S}^*$ is such that S functionally separates zero sets then $S = C^*(X)$.
- (d) If $S \in \mathfrak{S}^*$ is such that S generates the zero sets then $S = C^*(X)$.

PROOF. (a) \rightarrow (b). (a) \rightarrow (c). Since X is pseudocompact, zero sets satisfy the Isiwata property and their images under continuous functions on X also satisfy this property by Lemma B and hence are compact. One can also get this result as an immediate consequence of Isiwata's result (see [9]) that images of zero sets are closed subsets of $f(X)$, for X pseudocompact if each point of $f(X)$ is a G_δ . So if a function functionally separates disjoint zero sets it completely separates them, and thus by Theorem B, (a) \rightarrow (c). Also if a function f functionally separates a point x and a zero set Z , there will be a zero set H such that $x \in H \subset \sim Z$; hence f will completely separate them, so (a) \rightarrow (b). (b) \rightarrow (a). If X is not pseudocompact there is a one point extension of X , $Y = X \cup [y]$ in which $[y]$ is a zero set and in which X is C^* -embedded in Y . We construct a weaker topology on X , (X, \mathfrak{U}) by identifying y with a point of $x \in X$ and forming the quotient topology on X . Let A and B be disjoint zero sets of (X, \mathfrak{T}) . If $y \notin A'$, there exist functions f, g on Y to the real line such that $f^{-1}(0) = A \cup [y]$ and $g^{-1}(0) = A$ and $g(x) = g(y)$ so that if either $x \in A$ or $x \notin A$, A is a zero set of (X, \mathfrak{U}) . By the C^* -embedding of X in Y if $y \in A'$, then $y \notin B'$ so B is a zero set. Thus zero sets of (X, \mathfrak{T}) are functionally separated in (X, \mathfrak{U}) and hence so are points and zero sets. (c) \rightarrow (d) is immediate. (d) \rightarrow (a). If X is not pseudocompact, by Lemma 1 there is a z -embedding which is not a C^* -embedding into a space Y . Then $C^*(Y)$ generates the zero sets in X but $C^*(Y) \neq C^*(X)$.

In regard to condition (d), M. Jerison (1961) and Hager [6] showed that in a Lindelöf space a ring $S \in \mathfrak{S}$ that completely separates points and zero sets generates the zero sets. Since a set X is z -embedded in every compactification of itself iff X is Lindelöf or almost compact [3] we conclude

THEOREM 2. *The following are equivalent for a Tychonoff space X .*

- (a) X is Lindelöf or almost compact.
- (b) Any ring $S \in \mathfrak{S}^*$ that induces the topology on X generates the zero sets of $C^*(X)$.

THEOREM 3. *The following are equivalent for a Tychonoff space X .*

- (a) X is pseudocompact.
- (b) If $f \in S \in \mathfrak{S}^*$, and $Z(f) = \emptyset$ then $1/f \in S$.
- (c) If $f \in S \in \mathfrak{S}$, $|f| \in S$.
- (d) If $f, g \in S \in \mathfrak{S}$, then $\max(f, g) \in S$.

PROOF. (a) \rightarrow (b) was stated in [7]. We include a proof for completeness sake. Suppose $0 < |f| < n$. By the Isiwata property there is $\varepsilon > 0$ such that $0 < \varepsilon \leq |f| < n - \varepsilon$. Then for $f > 0$, $n/f = 1 + (1 - f/n) + (1 - f/n)^2 + \dots$, and for $f < 0$, $n/f = -1 - (1 + f/n) - (1 + f/n)^2 - \dots$ so that $1/f$ is a uniform limit of continuous functions of S . (b) \rightarrow (c), (c) \rightarrow (d) follow from well-known arguments. For instance, see Willard [15, §44]. (d) \rightarrow (c) is immediate. As noted by Hager, (b) \rightarrow (a) is immediate by considering $C^*(X)$ for X not pseudocompact. (c) \rightarrow (a) (based on an idea of W. W. Comfort). The function $|n|$ is not the uniform limit of polynomials on the integers I . If X is not pseudocompact, X contains a C -embedded copy of I , and so there is a continuous function $f \in S$ such that $|f| \notin S$ and S is generated by all extensions of polynomials on I .

Note by the method of proof the theorem is true if \mathfrak{S} is restricted to those rings that generate the topology on X .

3. Generalizations to functionally regular spaces.

DEFINITION 2. A space is functionally Hausdorff (regular) if points (points and closed sets) are functionally separated [1].

Stephenson [10]–[12] has made studies of functionally Hausdorff closed spaces and the question as to when these products are functionally Hausdorff closed. The spaces are precisely the spaces such that the weak topology is compact. The functionally regular closed spaces are precisely the functionally Hausdorff closed spaces that are functionally regular [1]. From the proof of Theorem 1, (a) \rightarrow (b) we have the following corollary.

COROLLARY 1. *The product of functionally regular closed spaces is functionally regular closed iff the product is pseudocompact.*

Unfortunately we do not know whether all such products are pseudocompact or if the theorem is true for functionally Hausdorff closed spaces. Stephenson [10] has an example of two functionally Hausdorff closed spaces such that their product is not functionally Hausdorff closed but the product is not pseudocompact either.

ADDED IN PROOF. Stephenson has recently shown that the products of functionally regular closed spaces are pseudocompact.

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