

A GENERALIZED POINCARÉ STABILITY CRITERION

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ABSTRACT. Let $\Phi_t^* \eta = \Phi^{-t} \circ \eta \circ \phi^t$ define a semigroup on the Banach space $\Gamma(M, E)$ of continuous sections of E over M . It is known that (Φ^t, ϕ^t) is hyperbolic iff Φ_t^* has spectrum off the unit circle for $t \neq 0$. We prove that a third equivalent condition is that the (unbounded!) infinitesimal generator L of $\{\Phi_t^*\}$ have its spectrum disjoint from the imaginary axis. In two dimensions this property coincides with the Poincaré stability criterion for a periodic orbit of a planar dynamical system.

1. Introduction. One may always associate, in a natural fashion, a linear semigroup with a given dynamical system. Thus, if a smooth vector bundle flow (Φ^t, ϕ^t) acts on a smooth Riemannian vector bundle (E, M, π) , we can define the one-parameter group Φ_t^* of bounded operators on the complex Banach space $\Gamma(E)$ of continuous sections η of E :

$$\Phi_t^* \eta = \Phi^{-t} \circ \eta \circ \phi^t.$$

Certain algebraic properties of this semigroup match up with corresponding qualitative features of the underlying vector bundle flow. The most important such feature is hyperbolicity. Recall that the pair (Φ^t, ϕ^t) is hyperbolic if E splits as a continuous Whitney sum of invariant subbundles $E = E^+ \oplus E^-$ such that Φ^t contracts E^+ and expands E^- . Equivalently [4], [6], (Φ^t, ϕ^t) is hyperbolic if Φ_t^* for any $t \neq 0$ is a hyperbolic operator; i.e., has spectrum, $\sigma(\Phi_t^*)$, disjoint from the unit circle. It is the goal of this paper to establish that such algebraic or spectral properties have their optimal expression as attributes of the infinitesimal generator of the induced semigroup. The generator of Φ_t^* is a closable operator L , densely defined in $\Gamma(E)$ by setting

$$L\eta = \frac{d}{dt} \Phi_t^* \eta \Big|_{t=0}.$$

We call an operator *infinitesimally hyperbolic* if its spectrum is disjoint from the imaginary axis. The principal result of this article is

THEOREM A. (Φ^t, ϕ^t) is hyperbolic if and only if L is infinitesimally hyperbolic.

In [1], we established a special case of Theorem A when (Φ^t, ϕ^t) is given by the tangent flow (Tf^t, f^t) of a nonsingular smooth flow f^t on M and the vector bundle E is the quotient TM/X with $X = df^t/dt$. Theorem A translates to the statement:

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the flow f' is Anosov if and only if the Lie derivative L_X is infinitesimally hyperbolic on $\Gamma(E)$.

For the tangent flow, if L_X is infinitesimally hyperbolic the periodic orbits are isolated; thus, the nonperiodic points are dense. This fact was decisive in our proof of the special case of Theorem A. However, a vector bundle flow can be hyperbolic when ϕ' has open sets of periodic points (Proposition 3.1). Because of this fact Theorem A requires a different proof in the vector bundle case which we present in §2.

Aside from its virtue of generality, the vector bundle formulation of Theorem A offers a true generalization of Poincaré's hyperbolicity criterion for a periodic orbit of a flow on the plane. Recall that if X is a vector field on \mathbf{R}^2 and Θ is a periodic orbit of the flow of X with period T , then Θ is hyperbolic exactly when

$$\frac{1}{T} \int_{\Theta} \operatorname{div}(X) \neq 0.$$

Now, the flow ϕ' of X defines a "Poincaré map" ρ from the orthogonal trajectory Σ_p at p on Θ to any other orthogonal trajectory Σ_q at $q = \phi'(p)$. If x is a local coordinate for Σ_p and y is a local coordinate for Σ_q with $x(p) = 0$ and $y(q) = 0$, then

$$\left. \frac{d\rho}{dx} \right|_{x=0} = \exp \left(\int_0^T \operatorname{div}(X)(\phi_s(p)) ds \right).$$

In our language this means that the tangent flow on the normal bundle over Θ is representable as the vector bundle flow F' on $S^1 \times \mathbf{R}$ given by

$$F'(\theta, v(\theta)) = \left(\theta + t, \exp \left(\int_{\theta}^{\theta+t} \operatorname{div}(X) ds \right) v(\theta) \right).$$

The infinitesimal generator L of F'_t is

$$L = \frac{d}{d\theta} - \operatorname{div}(X)(\theta).$$

To compute the spectrum of L , assume

$$\eta' - \operatorname{div}(X)\eta = \lambda\eta$$

for $\eta: S^1 \rightarrow \mathbf{R}$. Clearly,

$$\begin{aligned} \lambda &= \frac{1}{T} \int_0^T \eta' / \eta \, d\theta - \frac{1}{T} \int_0^T \operatorname{div}(X) \, d\theta \\ &= \frac{2\pi Ni}{T} - \frac{1}{T} \int_0^T \operatorname{div}(X) \, d\theta \end{aligned}$$

where $N \in \mathbf{Z}$. Since $\sigma(L)$ consists entirely of eigenvalues (L is elliptic) L is infinitesimally hyperbolic exactly when

$$\frac{1}{T} \int_0^T \operatorname{div}(X) \, d\theta \neq 0.$$

Hence, Theorem A yields Poincaré's criterion.

In both cases, hyperbolicity is implied by a condition, infinitesimal hyperbolicity, which does not require the integration of the vector field X (cf. [4, p. 8]). Of course,

this expresses the value of Poincaré's criterion and is one of the main motivations for proving Theorem A.

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2. Proof of Theorem A. The exponential image of the spectrum of the generator L is contained in the spectrum of the operator $\Phi^* = \Phi_1^*$ [3, p. 467]. Hence, a hyperbolic vector bundle flow induces a group of operators with an infinitesimally hyperbolic generator. To prove Theorem A we must show that an infinitesimally hyperbolic generator induces a hyperbolic group.

Let $x \in M$ and let E_x denote the fiber of E at x . Theorem A follows from the following proposition.

PROPOSITION 2.1. *If L is infinitesimally hyperbolic and v is a nonzero element of E_x then*

$$\sup\{|\Phi'_x v| \mid t \in \mathbf{R}\} = \infty.$$

In fact, from the general theory of semigroups [3, p. 467], the point spectrum and the residual spectrum of L exponentiate to give the point and residual spectra of Φ^* . Hence, if Φ^* fails to be hyperbolic there is a complex number $\lambda = e^{i\theta}$ on the unit circle which belongs to the approximate point spectrum $\sigma_{\text{ap}}(\Phi^*)$ of Φ^* ; i.e., a sequence of continuous sections η_k exists such that for each k

$$\|\eta_k\| = 1$$

and as $k \rightarrow \infty$,

$$\lim\|\Phi^* \eta_k - \lambda \eta_k\| = 0.$$

Choose $x_k \in M$ such that

$$|\eta_k(x_k)| = 1.$$

If $t \neq 0$ we have

$$|\Phi'_t \eta_k(x_k) - \lambda^t \eta_k(\phi^t(x_k))| \leq \|\Phi_t^* \eta_k - \lambda^t \eta_k\|.$$

Hence, as $k \rightarrow \infty$

$$\lim|\Phi'_t \eta_k(x_k) - \lambda^t \eta_k(\phi^t(x_k))| = 0.$$

Now, choose a vector v which is a limit point of the set $\{\eta_k(x_k)\}$ and observe that, choosing a subsequence if necessary, as $k \rightarrow \infty$

$$\lim \Phi'_t \eta_k(x_k) = \Phi'_x v.$$

Therefore,

$$|\Phi'_x v| \leq |\Phi'_x v - \Phi'_t \eta_k(x_k)| + |\Phi'_t \eta_k(x_k) - \lambda^t \eta_k(\phi^t(x_k))| + |\eta_k(x_k)|$$

and this implies that $|\Phi'_x v| \leq 2$. Since $t \neq 0$ was arbitrary, we have

$$\sup\{|\Phi'_x v| \mid t \in \mathbf{R}\} < \infty,$$

contrary to Proposition 2.1.

For the proof of Proposition 2.1 let BP^* denote the set of ϕ^t periodic points $x \in M$ for which there exists an open set U of periodic points and a number

$K > 0$ such that $\mathcal{O}_x \subset U$ and each $y \in U$ has prime period less than k . We show first that Proposition 2.1 holds when $v \neq 0$ is in E_x and $x \in BP$.

Assume that

$$\sup\{|\Phi'_x v| \mid t \in \mathbf{R}\} < \infty$$

and that the prime period of x is $T > 0$. Clearly, $\Phi_x^{-T}: E_x \rightarrow E_x$ is not hyperbolic. Hence, there is a vector $v \in E_x \oplus iE_x$ and $\theta \in \mathbf{R}$ such that

$$\Phi_x^{-T}v = e^{-i\theta T}v.$$

Define, for $0 < t \leq T$,

$$\xi(\phi^{-t}(x)) = e^{i\theta t}\Phi_x^{-t}v.$$

Then, ξ is a smooth section on \mathcal{O}_x and a computation shows

$$\Phi_T^*\xi = e^{i\theta T}\xi.$$

Again, from the general theory of semigroups [3, p. 467], since $e^{i\theta T}$ is in the point spectrum of Φ_T^* , the infinitesimal generator l of Φ_t^* on $\Gamma(\mathcal{O}_x)$ has an imaginary number $i\alpha$ in its point spectrum. In particular, there is a smooth section ξ_0 on \mathcal{O}_x such that $\|\xi_0\| = 1$ and $l\xi_0 = i\alpha\xi_0$.

Choose a smooth $\eta \in \Gamma(E)$ which agrees with ξ_0 on \mathcal{O}_x . As $x \in BP$, there is a neighborhood U of \mathcal{O}_x such that the prime period of any point $y \in U$ is less than some finite number K . Let d be a distance function compatible with the topology of M , and define, for $y \in U$

$$\delta(y) = \sup\{d(\phi^t(y), \mathcal{O}_x) \mid 0 < t < K\}.$$

Since, for fixed t ,

$$d(\phi^t(y), \mathcal{O}_x) \leq d(\phi^t(y), \phi^t(y_k)) + d(\phi^t(y_k), \mathcal{O}_x)$$

it follows that

$$|\delta(y) - \delta(y_k)| \leq \sup\{d(\phi^t(y), \phi^t(y_k)) \mid 0 < t < K\}.$$

Hence, since $[0, K]$ is compact, as $k \rightarrow \infty$

$$\lim \delta(y_k) = \delta(y),$$

i.e., δ is continuous in U . Choose $\varepsilon > 0$. As $L\eta - i\alpha\eta \in \Gamma(E)$ there is a neighborhood V of \mathcal{O}_x such that $V \subset U$ and

$$\sup\{|L\eta(x) - i\alpha\eta(x)| \mid x \in V\} < \varepsilon.$$

Finally, find $c > 0$ such that $W = \delta^{-1}([0, c]) \subset V$ and define

$$f(x) = 0, \quad x \notin W,$$

and

$$f(x) = 1 - c^{-1}\delta(x), \quad x \in W.$$

Then, f is a continuous ϕ^t -invariant function on M . Hence, $f\eta$ is in the domain of L and

$$\|Lf\eta - i\alpha f\eta\| \leq \|f\| \sup\{|L\eta(x) - i\alpha\eta(x)| \mid x \in W\} < \varepsilon.$$

This implies $i\alpha \in \sigma(L)$, contrary to the assumption that L is infinitesimally hyperbolic. The argument has been presented for periodic orbits with period

$T > 0$. However, if x is a rest point in BP and there is a $v \neq 0$ in E_x such that

$$\sup\{|\Phi'_x v| \mid t \in \mathbf{R}\} < \infty$$

a similar argument shows that L is not infinitesimally hyperbolic.

To complete the proof we must show Proposition 2.1 when $v \in E_x$, $v \neq 0$ and $x \in M \setminus BP$. Assume that there is a number $K > 0$ such that

$$\sup\{|\Phi'_x v| \mid t \in \mathbf{R}\} < K.$$

Observe that if $\|\Phi^*\| < 1$, then Φ^* is hyperbolic and Theorem A follows. Hence, we may assume $\|\Phi^*\| > 1$.

Choose $\varepsilon > 0$. We adapt an argument of Măné [5, p. 367] to produce a continuous section ξ such that $\|\xi\| = 1$ and $\|(I - \Phi^*)\xi\| < \varepsilon$. Pick $N \in \mathbf{Z}^+$ such that

$$N + 1 > 3K\|\Phi^*\|/\varepsilon\|v\|.$$

Since $x \in M \setminus BP$ we can find a point q near x with three properties:

(a) There is a neighborhood U of q such that $\phi^j(U) \cap U = \emptyset$ for j an integer with $|j| < N$,

(b) there is a vector $w \in E_q$ such that $|w| = |v|$ and $|\Phi'_q w| < 2K$ for $|j| < N$ and

(c) there is a smooth section $\eta \in \Gamma(E)$ with support in U such that $\eta(q) = w$,

$$|\Phi^j \eta(y)| < 3K \quad \text{for all } y \in U \text{ and } |j| < N.$$

Define $\xi \in \Gamma(E)$ by (here we correct an error in [5]):

$$\xi(y) = 0; \quad y \notin \bigcup_{|j| < N} \phi^j(U),$$

$$\xi(y) = (1 - |j|/(N + 1))\Phi^j \eta(\phi^{-j}(y)); \quad y \in \phi^j(U), |j| < N.$$

Then

$$\|(I - \Phi^*)\xi\| < \varepsilon\|\xi\|.$$

Therefore, a sequence $\eta_k \in \Gamma(E)$ exists such that

(a) $\|\eta_k\| = 1$,

(b) as $k \rightarrow \infty$, $\lim\|(\Phi^* - I)\eta_k\| = 0$ and

(c) for $N \in \mathbf{Z}^+$, there is an integer N_0 such that for each $k > N_0$ a point $q \in M$ exists which is nonperiodic or has prime period larger than N with $|\eta(q)| = 1$.

Choose $\varepsilon > 0$ and pick $N > (8c + \varepsilon)/\varepsilon$ where

$$c = \sup\{\|\Phi'_t\| \mid 0 \leq t < 1\}.$$

Then, there is a k so large that

(a) $\|\Phi^* \eta_k - \eta_k\| < \varepsilon/4c$ and

(b) a point x_0 and a cross-section Σ at x_0 exists such that

$$|\eta_k(x_0)| = 1$$

and $\Sigma \times [-N, N]$ is a flow box for ϕ^t .

With these choices, the construction in [1, Theorem 2.1] produces a smooth section $\eta \in \Gamma(E)$ such that

$$\|\eta\| = 1$$

and

$$\|L\eta\| \leq \varepsilon.$$

Hence, $0 \in \sigma(L)$ contrary to the assumption. This completes the proof of Proposition 2.1. Q.E.D.

3. Existence of hyperbolic flows. The following proposition shows that hyperbolic vector bundle flows exist over any given flow.

PROPOSITION 3.1. *If (F, M, ρ) is a vector bundle with compact base manifold M and ϕ' is a smooth flow on M , then there is a vector bundle flow (Ψ', ϕ') in (F, M, ρ) which is hyperbolic.*

PROOF. Choose a fiber metric g for F and let ∇ denote the associated metric connection. If $X = d\phi'/dt$ and I denotes the identity on F , define

$$A = \nabla_X + I.$$

A generates a strongly continuous group $T(t)$ of operators in $\Gamma(F)$. Since ∇_X and I commute, $T(t)$ has the form

$$T(t) = e^{tI}P_t = P_t e^{tI}$$

where for $\eta \in \Gamma(F)$, $P_t\eta(x)$ is the vector in E_x obtained by parallel transport of the vector $\eta(\phi'(x))$ along $\phi'(x)$ to x . Clearly, $T(t)$ also has the form $\Psi_t^\#$ for the vector bundle flow (Ψ', ϕ') where

$$\Psi'_x: E_x \rightarrow E_{\phi'(x)}$$

is the mapping "parallel transport along $\phi'(x)$ followed by e^{tI} ." Moreover, in terms of the norm in $\Gamma(F)$ induced by g , P_t is an isometry of $\Gamma(F)$. Since e^{tI} is hyperbolic and e^{tI} commutes with P_t , $\Psi_t^\#$ is hyperbolic. Q.E.D.

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