SIMPLE SUBRINGS OF ALGEBRAS OVER FIELDS

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ABSTRACT. In this note we shall prove that if A is a not necessarily associative algebra over a field K and R is a simple subring of A with centroid F then $\dim_F R < \dim_K A$.

Since we do not use polynomial identities in a proof of this result then we have obtained an affirmative answer to the 11th question from [2], posed by I. N. Herstein.

In this paper A denotes an algebra over a field K , R is a simple (idempotent) subring of A and F denotes a centroid of R . It is well known [1] that F is a field and R is an F -algebra.

THEOREM. $\dim F_R < \dim_K A$.

PROOF. Let $A_1 = KR$ be the K -subspace of A spanned by R . It is obvious that A_1 is a K -subalgebra of A containing R . Now let M be a maximal element of a family of ideals J of a ring A_1 such that $J \cap R = 0$. Put $B = A_1/M$. Since R is a simple ring then it is easy to see that M is a prime ideal of A_1 and consequently it is an ideal of the K -algebra A_1 . Thus B is a K -algebra containing R in a natural way. Now if I is a nonzero ideal of an algebra B then by the construction of B it follows that $I \cap R = R$. Now $R^2 = R$ and $B = KR$ implies that $I = B$, i.e. B is a simple algebra. Let L denote a centroid of B . Obviously we get the following inequalities

$$\dim_K A > \dim_K A_1 > \dim_K B > \dim_L B. \quad (1)$$

Let $f \in F$, $l_i \in L$, $r_i \in R$ and $s \in R$. Assume that $\sum l_i r_i = 0$. Then we have $(\sum l_i (f r_i))s = \sum l_i ((f r_i)s) = \sum l_i (r_i (f s)) = (\sum l_i r_i)(f s) = 0$. Then $(\sum l_i (f r_i))R = 0$. Since R generates B as a K -linear space then $(\sum l_i (f r_i))B = 0$. Analogously from the equality $s(\sum l_i (f r_i)) = (f s)(\sum l_i r_i) = 0$ it follows that $B(\sum l_i (f r_i)) = 0$. Since B is a simple ring then we see that $\sum l_i (f r_i) = 0$. Hence it follows that the rule $g(f) \cdot \sum l_i r_i = \sum l_i (f r_i)$ defines unambiguously the extension of action f on R to the action $g(f)$ on B . It is easily seen that g is an imbedding of F into L . In this way L and B become algebras over F .

Now we are able to define a natural homomorphism $h: L \otimes_F R \rightarrow B$ defined by the rule $h(\sum l_i \otimes r_i) = \sum l_i r_i$. Since h is a nonzero homomorphism and $L \otimes_F R$ is a simple ring (see [1, p. 292]) then h is an injection of $K := L$ algebras. Thus we have obtained

$$\dim_F R = \dim_L L \otimes_F R < \dim_L B. \quad (2)$$

Received by the editors March 26, 1980.

1980 *Mathematics Subject Classification*. Primary 16A56.

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0002-9939/81/0000-0151/\$01.50

This together with (1) ends the proof of the theorem.

Observe that by the proof of the theorem we get the following

COROLLARY. *If $\dim_K A = \dim_F R < \infty$ then A is a central simple K -algebra isomorphic to $K \otimes_F R$.*

REFERENCES

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