EXTENSION OF A THEOREM OF GUDDER AND SCHELP TO POLYNOMIALS OF ORTHOMODULAR LATTICES

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ABSTRACT. Consider a polynomial expression $p(b, c, \dots, d) = e$ where any two of the elements $b, c, \dots, d$ commute. If an element a commutes with e , then b commutes with $p(a, c, \dots, d)$.

1. Introduction. In this paper, $\mathcal{L} = (L, \vee, \wedge, ', 0, 1)$ always means an orthomodular lattice. If $a, b \in L$ and $a = (a \wedge b) \vee (a \wedge b')$, then a, b are said to commute; in this case we write aCb .

First recall some useful properties of the relation C (see e.g. [2, pp. 52–53]).

1.1. LEMMA. (i) If $a, b, c \in L$ are such that aCb and aCc , then $aCb \wedge c$ and $aCb \vee c$.

(ii) If $\{a, b, c\} = \{x, y, z\}$ is a subset of L , and if xCy and xCz , then $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

A well-known result states that if $a, b \in L$ and aCb , then bCa . For orthomodular lattices a result of Gudder and Schelp [5, p. 235] can be simplified and formulated in the following way: If $aCb \wedge c$ and bCc , then $bCa \wedge c$. The author's paper arose as the result of attempts to generalize these two special situations.

2. Preliminaries. In what follows, $\{b, c_2, \dots, c_n\}$ denotes a subset of L where any two elements commute.

2.1. LEMMA. Let p be an n -ary lattice polynomial. Then

(1) $p(b, c_2, \dots, c_n) = B_1(b, c_2, \dots, c_n) \wedge B_2(b, c_2, \dots, c_n)$, where

(2) $B_1(y_1, x_2, \dots, x_n) = \bigwedge (y_1 \vee \bigvee (x_i | i \in I_j) | j \in J)$,

(3) $B_2(y_1, x_2, \dots, x_n) = \bigwedge (\bigvee (x_k | k \in K_t) | t \in T)$

and where I_j, J, K_t, T are finite sets.

PROOF. By Lemma 1(ii), the sublattice generated by $b, c_2, \dots, c_n$ is distributive and the assertion follows from [1, Theorem 12, p. 145].

2.2. COROLLARY. Let $e_1 = d, e_2, \dots, e_n \in L$ be such that e_iCe_j for $1 \leq i, j \leq n$ and suppose (1) is valid. Then

$$p(d, e_2, \dots, e_n) = B_1(d, e_2, \dots, e_n) \wedge B_2(d, e_2, \dots, e_n)$$

where B_1 and B_2 are the same polynomials as in (2) and (3).

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2.3. PROPOSITION. *If $a, b \in L$ are such that $aCb \wedge c$ and bCc , then $bCa \wedge c$.*

PROOF. Since $a = [a \wedge (b \wedge c)] \vee [a \wedge (b \wedge c)']$, we have

$$a' = (a' \vee b' \vee c') \wedge [a' \vee (b \wedge c)].$$

Thus

$$a' \vee c' = \{(a' \vee b' \vee c') \wedge [a' \vee (b \wedge c)]\} \vee c'.$$

However, $a' \vee b' \vee c' \geq a \wedge (b' \vee c') = [a' \vee (b \wedge c)]'$ and it follows $a' \vee (b \wedge c) \leq C a' \vee b' \vee c'$. Similarly, $c' \leq C a' \vee b' \vee c'$. By Lemma 1.1(ii),

$$a' \vee c' = (a' \vee b' \vee c') \wedge [a' \vee c' \vee (b \wedge c)].$$

But $c' \vee (b \wedge c) = c' \vee b$, by Lemma 1.1(ii). Therefore

$$a' \vee c' = (a' \vee c' \vee b') \wedge (a' \vee c' \vee b)$$

and, finally, $a \wedge c = [(a \wedge c) \wedge b] \vee [(a \wedge c) \wedge b']$.

3. Key lemma and main theorem. We begin this section with the following technical lemma.

3.1. LEMMA. *Let p be an n -ary lattice polynomial and let (1) be valid. Then for every $a \in L$*

$$D_1 \wedge D_2 \leq p(a, c_2, \dots, c_n) \leq D_2$$

where

$$D_1 = \bigwedge (\bigvee (c_i | i \in I_j) | j \in J),$$

$$D_2 = \bigwedge (\bigvee (c_k | k \in K_t) | t \in T).$$

PROOF. Since p is a lattice polynomial, by [3, Lemma 6, p. 33] we have

$$p(0, c_2, \dots, c_n) \leq p(a, c_2, \dots, c_n) \leq p(1, c_2, \dots, c_n).$$

By Corollary 2.2,

$$p(0, c_2, \dots, c_n) = B_1(0, c_2, \dots, c_n) \wedge B_2(0, c_2, \dots, c_n) = D_1 \wedge D_2.$$

Moreover,

$$p(1, c_2, \dots, c_n) = B_1(1, c_2, \dots, c_n) \wedge B_2(1, c_2, \dots, c_n) = D_2.$$

3.2. THEOREM. *Let $c_1, c_2, \dots, c_n$ and a be elements of an orthomodular lattice. If $c_i C c_j$ for any $1 \leq i, j \leq n$ and if $a C p(c_1, c_2, \dots, c_n)$ where p is an n -ary lattice polynomial, then*

$$c_k C p(c_1, c_2, \dots, c_{k-1}, a, c_{k+1}, \dots, c_n)$$

for every $k = 1, 2, \dots, n$.

PROOF. We may assume that $k = 1$. Let $b = c_1$. By assumption, $a C p(b, c_2, \dots, c_n)$. From Lemma 1.1(i), $c_i C p(b, c_2, \dots, c_n)$ for every $i = 2, \dots, n$. By the same lemma we also have $p(a, c_2, \dots, c_n) C p(b, c_2, \dots, c_n)$. That is,

$$B_1(b, c_2, \dots, c_n) \wedge B_2(b, c_2, \dots, c_n) C p(a, c_2, \dots, c_n).$$

Since $b C c_i$ for every $i = 2, \dots, n$,

$$B_1(b, c_2, \dots, c_n) = b \vee (\bigwedge (\bigvee (c_i | i \in I_j) | j \in J)) = b \vee D_1.$$

Replacing $B_2(b, c_2, \dots, c_n)$ by D_2 we see that

$$(b \vee D_1) \wedge D_2 Cp(a, c_2, \dots, c_n).$$

By Lemma 1.1(ii), bCD_1 and bCD_2 . Hence $(b \wedge D_2) \vee (D_1 \wedge D_2)Cp(a, c_2, \dots, c_n)$ and also $b \wedge D_2CD_1 \wedge D_2$. In view of the assertion dual to Proposition 2.3, $b \wedge D_2Cp(a, c_2, \dots, c_n) \vee (D_1 \wedge D_2)$. Then, by Lemma 3.1, $b \wedge D_2Cp(a, c_2, \dots, c_n)$. Similarly, $bCp(a, c_2, \dots, c_n) \wedge D_2 = p(a, c_2, \dots, c_n)$.

FIGURE 1

REMARK. We conclude by observing that the assertion of Theorem 3.2 is optimal in the following sense: If q is a polynomial in $\vee$, $\wedge$ and $'$, then an analogue of Theorem 3.2 need not be valid. This can easily be seen by considering the orthomodular lattice constructed by Greechie's classical method [4] from two copies of 2^3 (cf. Figure 1). Let $q(x, y) = (x \wedge y) \vee x'$. Here we have $aCq(d', e) = 1$ but d' and $q(a, e) = b$ do not commute.

REFERENCES

1. G. Birkhoff, *Lattice theory*, 2nd ed., Amer. Math. Soc. Colloq. Publ., Amer. Math. Soc., Providence, R.I., 1948. MR 10, p. 673.
2. ———, *Lattice theory*, 3rd ed., Amer. Math. Soc. Colloq. Publ., vol. 25, Amer. Math. Soc., Providence, R.I., 1967. MR 37 #2638.
3. G. Grätzer, *Lattice theory: First concepts and distributive lattices*, Freeman, San Francisco, Calif., 1971. MR 48 #184.
4. R. J. Greechie, *On the structure of orthomodular lattices satisfying the chain condition*, J. Combin. Theory 4 (1968), 210–218. MR 37 #2641.
5. S. P. Gudder and R. H. Schelp, *Coordinatization of orthocomplemented and orthomodular posets*, Proc. Amer. Math. Soc. 25 (1970), 229–237. MR 41 #3336.

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