NEIGHBORHOODS OF UNIVALENT FUNCTIONS

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ABSTRACT. For an analytic function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ in the unit disc E conditions are established such that all functions $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in N_{\delta}(f)$, i.e. $\sum_{k=2}^{\infty} k|a_k - b_k| < \delta$, are in some class of univalent functions in E . For instance, we prove that every $g \in N_{1/4}(f)$ is starlike univalent in E if f is convex univalent.

Let A denote the class of analytic functions f in the unit disc $E = \{z \mid |z| < 1\}$ with $f(0) = 0$, $f'(0) = 1$. For $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A$ and $\delta > 0$ we define the neighborhood $N_{\delta}(f)$ as follows.

$$N_{\delta}(f) = \left\{ g(z) = z + \sum_{k=2}^{\infty} b_k z^k \mid \sum_{k=2}^{\infty} k|a_k - b_k| < \delta \right\}.$$

For the identity function $e(z) = z$ we have (A. W. Goodman [2])

$$N_1(e) \subset S^*, \quad (1)$$

where $S^* \subset A$ denotes the class of normalized starlike functions w.r.t. the origin. The following simple proof is due to J. Clunie and F. R. Keogh [1]. For

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in N_1(e)$$

we have

$$|zg'(z) - g(z)| < \sum_{k=2}^{\infty} (k-1)|b_k| |z|^k < |z| - \sum_{k=2}^{\infty} |b_k| |z|^k < |g(z)|$$

and thus $|zg'(z)/g(z) - 1| < 1$ in E . This shows $g \in S^*$.

Corresponding problems for other functions have apparently not been studied so far, probably because of technical difficulties. In the present paper we shall apply simple convolution methods to fill this gap. We recall that the convolution (Hadamard product) of two functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

is given by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

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Besides the general Theorems 1 and 2 below, our main result is the following extension of (1).

COROLLARY 1. Let $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$ be convex univalent in E . Then

$$N_{\delta_n}(f) \subset S^*, \quad \delta_n = 2^{-2/n}. \quad (2)$$

The values for δ_n are best possible w.r.t. the whole class under consideration.

Clearly (2) implies (1) for $n \rightarrow \infty$.

Besides S^* we shall deal with the following subclasses of A .

(i) S is the class of univalent functions in A .

(ii) M contains those functions $f \in A$ with

$$\frac{1}{z}(f * g)(z) \neq 0, \quad z \in E, \quad (3)$$

for all $g \in S^*$.

(iii) K is the class of close-to-convex functions, i.e. $f \in K$ if and only if $\operatorname{Re}[e^{i\varphi} z f'(z)/g(z)] > 0$ in E for certain $\varphi \in \mathbf{R}$, $g \in S^*$.

(iv) S_α^* , $0 < \alpha < 1$, contains the functions starlike of order α , i.e. $\operatorname{Re}[z f'(z)/f(z)] > \alpha$, $z \in E$. In particular $S^* = S_0^*$.

(v) C is the class of convex univalent functions in A .

(vi) $\operatorname{Sp}(\beta)$, $|\beta| < \pi/2$, is the class of β -spirallike functions f , i.e. $\operatorname{Re}[e^{i\beta} z f'(z)/f(z)] > 0$ in E .

It is well known that $C \subset S_{1/2}^* \subset S^* \subset K \subset M \subset S$ and $\operatorname{Sp}(\beta) \subset S$.

THEOREM 1. (i) Let $f \in A$. Then we have $N_\delta(f) \subset S$ at most for

$$\delta = \inf\{|f'(z)| \mid z \in E\}.$$

(ii) Let Q be one of the classes S , M , S^* . Let $f \in A$ and for all $\varepsilon \in \mathbf{C}$, $|\varepsilon| < \delta$, assume

$$\frac{f(z) + \varepsilon z}{1 + \varepsilon} \in Q. \quad (4)$$

Then $N_\delta(f) \subset Q$.

REMARKS. (1) Theorem 1 with $f = e$ and $Q = S^*$ gives again (1). (2) Although the bounds for δ given in Theorem 1 are not sharp for every f they are sharp in special cases. For instance, the functions $z + \mu z^2$ are in S^* if and only if $|\mu| < \frac{1}{2}$. Hence $f(z) = z + z^2/4$ fulfils (4) for $|\varepsilon| < \frac{1}{2}$, $Q = S^*$. For $\delta > \frac{1}{2}$ we have $g(z) = z + (1 + 2\delta)z^2/4 \in N_\delta(f)$ but $g \notin S^*$. This shows that $\delta = \frac{1}{2}$ is best possible in this case.

THEOREM 2. Let $f \in A$, $\beta \in \mathbf{R}$ with $|\beta| < \pi/2$. For all $\varepsilon \in \mathbf{C}$, $|\varepsilon| < \delta$, assume

$$\frac{f(z) + \varepsilon z}{1 + \varepsilon} \in \operatorname{Sp}(\beta).$$

Then $N_\gamma(f) \subset \operatorname{Sp}(\beta)$ where $\gamma = \delta \cos \beta$.

This result is not sharp for any $f \in \operatorname{Sp}(\beta)$, $\beta \neq 0$, except for $\delta = 0$. It cannot be improved, however, as shown by an investigation of functions of the form $z + \mu_n z^n$, $n \rightarrow \infty$.

Besides Corollary 1 we wish to mention another simple application of Theorem 1.

COROLLARY 2. *Let Q be one of the classes S, M, S^* . Let $f \in A$ be such that $F(z) = f(\gamma z)/\gamma \in Q$ for a certain $\gamma > 1$. Then $N_\delta(f) \subset Q$ with*

$$\delta = \gamma^2(\gamma - 1)(\gamma + 1)^{-3}.$$

In Corollary 1, $n = 1$, we study the full class C which fulfils $C \subset S_{1/2}^*$. In fact, $S_{1/2}^*$ shares many properties with C and it is a natural question whether Corollary 1 extends to $S_{1/2}^*$. The somewhat surprising answer is in Theorem 3.

THEOREM 3. (i) *For $f \in S_{1/2}^*$ we have $N_{1/4}(f) \subset K$.*

(ii) *For no $\alpha \in [0, 1)$ there is a positive δ such that $N_\delta(S_\alpha^*) \subset S^*$.*

Proofs.

PROOF OF THEOREM 1. (i) Let $\delta = \inf\{|f'(z)| \mid z \in E\}$. For any $\delta' > \delta$ we can choose $z_0 \in E$ such that $|f'(z_0)/z_0| < \delta'$. Then $g(z) = f(z) - f'(z_0)z^2/(2z_0) \in N_\delta(f)$ and $g'(z_0) = 0$. Hence $g \notin S$.

(ii) Let

$$Q' = \begin{cases} \left\{ \frac{z}{(1-xz)(1-yz)} \mid |x| < 1, |y| < 1 \right\}, & Q = S, \\ \left\{ \frac{1}{1+i\alpha} \left[\frac{z}{(1-z)^2} + i\alpha \frac{z}{1-z} \right] \mid \alpha \in \mathbb{R} \right\}, & Q = S^*, \\ S^*, & Q = M. \end{cases}$$

From the definition (or by a simple calculation if $Q = S^*$) it follows that for $f \in A$

$$f \in Q \Leftrightarrow \frac{1}{z}(h * f)(z) \neq 0, \quad z \in E, \text{ for all } h \in Q'. \quad (5)$$

Further we note

$$h(z) = z + \sum_{k=2}^{\infty} c_k z^k \in Q' \Rightarrow |c_k| < k, \quad k = 2, 3, \dots$$

Now let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A$ fulfil (4) for all $|\varepsilon| < \delta$. Then, by (5), we obtain for $h \in Q'$

$$\frac{1}{z} \frac{(f * h)(z) + \varepsilon z}{1 + \varepsilon} \neq 0, \quad z \in E, |\varepsilon| < \delta,$$

or

$$\left| \frac{(f * h)(z)}{z} \right| > \delta, \quad z \in E. \quad (6)$$

We notice that (6) is equivalent to (4). Assume $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in N_\delta(f)$. Then for $h \in Q'$ we have

$$\left| \frac{1}{z}(g * h) \right| = \left| \frac{f * h}{z} + \frac{(g - f) * h}{z} \right| > \delta - \left| \frac{(g - f) * h}{z} \right|.$$

But for $z \in E$

$$\left| \frac{(g-f) * h}{z} \right| = \left| \sum_{k=2}^{\infty} (b_k - a_k) c_k z^k \right| < |z| \sum_{k=2}^{\infty} k |b_k - a_k| < \delta, \quad (7)$$

and therefore $(g * h)/z \neq 0$ in E for all $h \in Q'$. From (5) we obtain the desired conclusion.

PROOF OF THEOREM 2. If we put $Q = \text{Sp}(\beta)$ then (5) with

$$Q' = \left\{ \frac{1}{e^{i\beta} + i\alpha} \left[e^{i\beta} \frac{z}{(1-z)^2} + i\alpha \frac{z}{1-z} \right] \mid \alpha \in \mathbb{R} \right\}$$

holds also for this case. Here we have

$$h(z) = z + \sum_{k=2}^{\infty} c_k z^k \in Q' \Rightarrow |c_k| < k/\cos \beta, \quad k = 2, 3, \dots$$

Now we follow the lines of the proof of Theorem 1 where we only have to replace $g \in N_\delta(f)$ by $g \in N_\gamma(f)$ and (7) by

$$\left| \frac{(g-f) * h}{z} \right| < |z| \sum_{k=2}^{\infty} \frac{k}{\cos \beta} |b_k - a_k| < \gamma/\cos \beta = \delta.$$

For the proof of Corollary 1 we need some preparation.

LEMMA 1. Let $w(z)$ be analytic in E , $w^{(k)}(0) = 0$ for $0 < k < n$. Let $z_0 \in E$ be such that $|w(z)| < |w(z_0)|$ for all $|z| < |z_0|$. Then $\kappa = z_0 w'(z_0)/w(z_0)$ is real and $> n$.

Lemma 1 is an extended version of what is called the Clunie-Jack Lemma (see I. S. Jack [3]). We take the opportunity to remark that this result is an immediate consequence of the classical Julia's Theorem [4]: let $q(z)$ be analytic in E and in a neighborhood of $z = 1$. Assume $|q(z)| < 1$ in E , $q(1) = 1$. Then

$$\frac{|1 - q(z)|^2}{1 - |q(z)|^2} < q'(1) \frac{|1 - z|^2}{1 - |z|^2}.$$

In particular, $q'(1) > 1$ if $q(0) = 0$. To prove Lemma 1 we put

$$q(z) = \frac{w(z_0 z)}{w(z_0) z^{n-1}}.$$

The relation $q'(1) = \kappa - (n - 1)$ gives the result.

The following is an application of Lemma 1. For $n > 1$ and

$$g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k \in S^*,$$

we have

$$|g(z)| \geq \frac{|z|}{(1 + |z|^n)^{2/n}}, \quad z \in E. \quad (8)$$

In fact, the function $w(z)$ defined by

$$\left(\frac{g(z)}{z} \right)^{n/2} = \frac{1}{1 - w(z)}$$

is analytic in E with $w^{(k)}(0) = 0$, $0 < k < n - 1$.

Furthermore, $|w(z)| < 1$ in E because otherwise there would be a $z_0 \in E$ with $|w(z_0)| = 1$ to which Lemma 1 applies. Then, by differentiation,

$$\frac{n}{2} \frac{z_0 g'(z_0)}{g(z_0)} = \frac{z_0 w'(z_0)}{w(z_0)} \frac{w(z_0)}{1 - w(z_0)} + \frac{n}{2}$$

and $\operatorname{Re} z_0 g'(z_0)/g(z_0) < 0$ which contradicts the starlikeness of $g(z)$. Thus, by Schwarz' Lemma, $|w(z)| < |z|^n$ in E and (8) follows immediately.

LEMMA 2. Let $s(z) = z + \sum_{k=-n+1}^{\infty} a_k z^k \in K$ such that $\operatorname{Re}[e^{i\varphi} z s'(z)/g(z)] > 0$ for a certain $g(z) = z + \sum_{k=-n+1}^{\infty} b_k z^k \in S^*$ and $\varphi \in \mathbb{R}$. Then we have $|s(z)/z| \geq 2^{-2/n}$.

PROOF. An application of Schwarz' Lemma gives

$$\frac{zs'(z)}{g(z)} = \frac{1 + xw(z)}{1 + yw(z)} \quad (9)$$

for certain $|x| = |y| = 1$ and an analytic function $w(z)$ with $|w(z)| < |z|^n$. (8) and (9) lead to

$$|s'(z)| \geq \frac{1 - |z|^n}{(1 + |z|^n)^{1+2/n}}, \quad z \in E.$$

Since $s(z)$ is univalent this estimate can be integrated to give the result as $|z| \rightarrow 1$.

PROOF OF COROLLARY 1. Let $Q = S^*$. Since $Q' \subset K$ we can confine ourselves to show (see (6))

$$|(f * h)(z)/z| \geq 2^{-2/n} \quad (10)$$

for $f(z) = z + \sum_{k=-n+1}^{\infty} a_k z^k \in C$ and $h \in K$. The following results have been established in [6].

(i) $f \in C, g \in S^* \Rightarrow f * g \in S^*$,

(ii) $f \in C, g \in S^*, h \in K$ and $\operatorname{Re}[e^{i\varphi} z h'(z)/g(z)] > 0$ implies $\operatorname{Re}[e^{i\varphi} z (f * h)'/(f * g)] > 0$ in E .

This shows that $f * h$ of (10) is close-to-convex of that type described in Lemma 2, which therefore settles (10). The sharpness of the result can be demonstrated using

$$f(z) = \int_0^z (1 + t^n)^{-2/n} dt$$

and $g(z) = f(z) - (\delta_n/2)z^2 \in N_{\delta_n}(f)$ which satisfies $g'(1) = 0$.

PROOF OF COROLLARY 2. (i) Let $Q = S$. Then for $h(z) = z/((1 - xz)(1 - yz)) \in Q', z \in E$, an application of Goluzin's inequality (see [5, p. 64]) to $F \in S$ gives

$$\left| \frac{(f * h)(z)}{z} \right| = \left| \frac{F(xz/\gamma) - F(yz/\gamma)}{(x - y)z/\gamma} \right| \geq \gamma^2(\gamma - 1)(\gamma + 1)^{-3}.$$

This shows that (4) is fulfilled for the δ given in the corollary.

(ii) Let $Q = M$. It follows from the results in [2] that for all $h \in Q' = S^*$ we have $h * F = g'$ for a certain $g \in M$. Therefore, for $z \in E$,

$$\left| \frac{(f * h)(z)}{z} \right| = \left| \frac{(F * h)(z/\gamma)}{z/\gamma} \right| = |g'(z/\gamma)| \geq \gamma^2(\gamma - 1)(1 + \gamma)^{-3}. \quad (11)$$

Again the conclusion follows from the equivalence of (6) and (4). If $Q = S^*$ we have $Q' \subset M$ and (11) remains valid also in this case.

PROOF OF THEOREM 3. (i) For $f \in S_{1/2}^*$ the relations $\operatorname{Re} f(z)/z > \frac{1}{2}$ and thus $|z/f(z)| < 2$ are well known. For $g \in N_{1/4}(f)$ we have $|f'(z) - g'(z)| \leq \frac{1}{4}$, such that in E

$$\begin{aligned} \operatorname{Re} \frac{zg'(z)}{f(z)} &= \operatorname{Re} \frac{zf'(z)}{f(z)} + \operatorname{Re} \frac{zg'(z) - zf'(z)}{f(z)} \\ &> \frac{1}{2} - \left| \frac{z}{f(z)} \right| |g'(z) - f'(z)| > 0. \end{aligned}$$

Hence $g \in K$.

(ii) Let $Q = S^*$. We shall show that

$$\inf \left| \frac{(f * h)(z)}{z} \right| = 0, \quad (12)$$

where the infimum is to be taken over $z \in E$, $f \in S_\alpha^*$ and $h \in Q'$. In fact, for f and z fixed we have

$$L(f, z) = \min_{h \in Q'} \left| \frac{(f * h)(z)}{z} \right| = \frac{1}{2} \left| \left| f'(z) - \frac{f(z)}{z} \right| - \left| f'(z) + \frac{f(z)}{z} \right| \right|.$$

For $f_\alpha = z(1 - z^2)^{1-\alpha}$ a direct calculation gives

$$L(f_\alpha, e^{i\varphi}) = \varphi^\alpha \left(\frac{\alpha 2^{\alpha-1}}{1-\alpha} + o(1) \right), \quad \varphi \rightarrow +0, \alpha \in (0, 1).$$

By continuity arguments, (12) follows. Now let $\delta > 0$. We choose $f \in S_\alpha^*$, $h(z) \in Q'$ and $z_0 \in E$ such that for $\mu = (f * h)(z_0)/z_0^2$ we have $|\mu| < \delta/2$. Let $g(z) = f(z) - (2\mu/h''(0))z^2$. Since $|h''(0)| = 2|(\alpha + 2)/(\alpha + 1)| > 2$ we have $g \in N_\delta(f)$ and

$$\frac{(g * h)(z_0)}{z_0} = \frac{(f * h)(z_0)}{z_0} - \mu z_0 = 0$$

which contradicts $g \in S^*$.

Open problems. (1) It would be interesting to know whether Theorem 1 holds also for $Q = K$, the set of close-to-convex functions.

(2) In Corollary 1 we have seen that $N_{1/4}(C) \subset S^*$. In fact, this is a very small subset of S^* . How can it be characterized geometrically?

(3) In the introduction we have seen that the functions $g \in N_1(e)$ are not only starlike but fulfil the stronger condition

$$\left| \frac{zg'(z)}{g(z)} - 1 \right| < 1, \quad z \in E. \quad (13)$$

Let T be the class of functions $g \in A$ which satisfy (13). One can show that Theorem 1 holds also for $Q = T$ so that for $f = e$ we in fact obtain the sharp result $N_1(e) \subset T$ also by our method. It would be of interest to find an analogue to Corollary 1 for T instead of S^* . Clearly we have to replace C by some subclass and a natural candidate would be the class of functions $f \in A$ with

$$\left| \frac{zf''(z)}{f'(z)} \right| < 1, \quad z \in E.$$

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