ESSENTIAL NUMERICAL RANGE IN $B(l_1)$

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ABSTRACT. In recent years, the numerical range lifting problem has been solved for operators on l_p , $1 < p < \infty$, and on certain Orlicz spaces l_M . Specifically, given an operator A , there exists a compact perturbation $A + C$ such that the numerical range of $A + C$ equals the essential numerical range of A . This result has also been established for essentially Hermitian operators on l_1 . In the present paper, the authors establish this result for all operators on l_1 .

1. Introduction. Let $B(X)$ and $C(X)$ denote, respectively, the algebras of bounded and compact linear operators on a Banach space X . The quotient algebra $B(X)/C(X)$ is called the Calkin algebra. A general problem which has been receiving much attention lately is: if a residue class $A + C(X)$ in the Calkin algebra possesses a certain property, does there exist a representative $A + C$ possessing the same property? For example, if X is a separable Hilbert space, then for each $A \in B(X)$ there exists $C \in C(X)$ such that the spectrum of $A + C$ equals the Weyl spectrum of A . See [11]. In this paper we are concerned with the following problem: for each $A \in B(X)$, does there exist $C \in C(X)$ such that the numerical range of $A + C$ equals the essential numerical range of A ? This question is answered in the affirmative for $X = l_p$, $1 < p < \infty$ (see [1], [6], [7]) and for $X = l_1$ in the case where the essential numerical range of A has no interior points. See [8]. The techniques used in [7] do not apply to $X = l_1$ to solve the problem if there are interior points in the essential numerical range of A since $C(l_1)$ is not an M -ideal in $B(l_1)$. See [10]. In this paper we use a straightforward computational technique to settle this question in the affirmative for $X = l_1$ for general $A \in B(l_1)$.

2. If A is a Banach algebra with unit, and $x \in A$, then $V(x) = \{s(x) : s \text{ is a normalized state in } A^*\}$ is called the numerical range of x . Hence for $A \in B(l_1)$, let $V(A)$ be the numerical range of A considered as an element of the Banach algebra $B(l_1)$, and let $V \text{ ess}(A)$ be the numerical range of $A + C(l_1)$ considered as an element of the Banach algebra $B(l_1)/C(l_1)$. $V \text{ ess}(A)$ is called the essential numerical range of A . See [4] and [5].

A linear operator A on l_1 can be represented by an infinite matrix $\{a_{ij}\}$. Let $C_j = \sum_{i=1}^{\infty} |a_{ij}|$. Then A is bounded if and only if $\sup_j C_j < \infty$ and $\|A\| = \sup_j C_j$. Also,

$$\|A\|_e = \lim_{n \rightarrow \infty} \sup_j \sum_{i=n}^{\infty} |a_{ij}| = \|A + C(l_1)\|.$$

Received by the editors November 11, 1979 and, in revised form, April 11, 1980; presented to the Society, January 5, 1980.

1980 *Mathematics Subject Classification*. Primary 47A12; Secondary 47A55, 47B37.

Key words and phrases. Essential numerical range, Calkin algebra, lifting problems, M -ideal.

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0002-9939/81/0000-0158/\$02.25

Hence A is compact if and only if $\lim_{n \rightarrow \infty} \sup_j \sum_{i=n}^{\infty} |a_{ij}| = 0$. See [13] and [10].

LEMMA 1. *Let $A \in B(l_1)$ with $\text{diag } A = 0$. Then there exists a compact perturbation A' of A , with $\text{diag } A' = 0$, such that $C'_j = \|A'\|_e$ for $j = 1, 2, \dots$*

PROOF. By [9], there is a compact perturbation A'' of A with $\|A''\| = \|A''\|_e$. It is clear from this construction that $\text{diag } A'' = 0$. Hence $C'_j \leq \|A''\|_e$ for all j . Now let A' be the result of perturbing A'' in only the first or second row, without bothering $\text{diag } A''$, such that $C'_j = \|A'\|_e$. A' is clearly a compact perturbation of A .

LEMMA 2. *If $A \in B(l_1)$, then there exists a compact perturbation A' of A such that the sequence of diagonal elements of A' is dense in itself.*

PROOF. Let $\{a_{ij}\}$ be the matrix representation of A , and let L be the set of cluster points of the sequence $\{a_{jj}\}$, $j = 1, 2, \dots$. Define a'_{jj} to be a nearest point to a_{jj} from L . Then $a_{jj} - a'_{jj} \rightarrow 0$ as $j \rightarrow \infty$. To see this, suppose to the contrary that there is an $\epsilon > 0$ and $j_n \rightarrow \infty$ such that $|a_{j_n j_n} - a'_{j_n j_n}| > \epsilon$. Since $\{a_{jj}\}$ is bounded by $\|A\|$, there is a subsequence $\{a_{j_n k j_n k}\}$ which has a limit, say l . Then $l \in L$ and so for sufficiently large k , there is a point in L closer to $a_{j_n k j_n k}$ than $a'_{j_n k j_n k}$, which is a contradiction. Hence define A' to be the operator represented by the same matrix that represents A , with a_{jj} replaced by a'_{jj} . Then A' is a compact perturbation of A . If $a'_{j_0 j_0}$ is a diagonal element of A' , then by definition, $a'_{j_0 j_0} \in L$. Hence there is a sequence $j_n \rightarrow \infty$ such that $a_{j_n j_n} \rightarrow a'_{j_0 j_0}$. Now $|a_{j_n j_n} - a'_{j_n j_n}| \leq |a_{j_n j_n} - a'_{j_0 j_0}|$, and so

$$\begin{aligned} |a'_{j_n j_n} - a'_{j_0 j_0}| &\leq |a'_{j_n j_n} - a_{j_n j_n}| + |a_{j_n j_n} - a'_{j_0 j_0}| \\ &\leq 2|a_{j_n j_n} - a'_{j_0 j_0}| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Hence the sequence of diagonal elements of A' is dense in itself.

The following theorem is the key to the main result, and is of independent interest since it provides a best compact approximate to an infinite set of operators simultaneously.

THEOREM 3. *If $A \in B(l_1)$, then there exists $C \in C(l_1)$ such that $\|A + C + \lambda I\| = \|A + \lambda I\|_e$ for all $\lambda \in \mathbb{C}$.*

PROOF. By Lemmas 1 and 2 we can assume that $\sum_{i \neq j} |a_{ij}| = \|A - \text{diag } A\|_e$ for each $j = 1, 2, \dots$ and that the sequence $\{a_{jj}\}$ is dense in itself.

Now for $\lambda \in \mathbb{C}$, let

$$L(\lambda) = \lim_{n \rightarrow \infty} \sup_{j \geq n} \left[\sum_{\substack{i=n \\ i \neq j}}^{\infty} |a_{ij}| + |a_{jj} + \lambda| \right] = \|A + \lambda I\|_e$$

and define $d_j = \sup_{\lambda} [\|A - \text{diag } A\|_e + |a_{jj} + \lambda| - L(\lambda)]$. Since the sequence $\{a_{jj}\}$ is dense in itself, it is easy to see that $|a_{jj} + \lambda| \leq L(\lambda)$ for each $\lambda \in \mathbb{C}$ and each $j = 1, 2, \dots$. Hence $d_j \leq \|A - \text{diag } A\|_e$ for each $j = 1, 2, \dots$. Let $c_j = \max\{d_j, 0\}$. We now define C to be the operator represented by the matrix $\{c_{ij}\}$, where $c_{jj} = 0$ and c_{ij} , $i \neq j$, are chosen as follows: if $c_j = \|A - \text{diag } A\|_e$, let

$c_{ij} = -a_{ij}$ for all $i > 0$. If $c_j < \|A - \text{diag } A\|_e$, let

$$c_{ij} = \begin{cases} -a_{ij} & \text{for } i = 1, \dots, m_j - 1, \\ c & \text{for } i = m_j, \\ 0 & \text{for } i = m_j + 1, \dots, \end{cases}$$

where m_j is defined by $\sum_{i=1}^{m_j-1} |a_{ij}| < c_j$ and $\sum_{i=1}^{m_j} |a_{ij}| > c_j$. The number c above is chosen so that $\sum_{i=1}^{\infty} |c_{ij}| = c_j$ and $\sum_{i \neq j} |a_{ij} + c_{ij}| = \|A - \text{diag } A\|_e - c_j$. Hence for all j , we have $\sum_{i=1}^{\infty} |c_{ij}| = c_j$ and $\sum_{i \neq j} |a_{ij} + c_{ij}| = \|A - \text{diag } A\|_e - c_j$.

Now we have

$$\begin{aligned} \sum_{i \neq j} |a_{ij} + c_{ij}| + |a_{jj} + \lambda| &= \|A - \text{diag } A\|_e - c_j + |a_{jj} + \lambda| \\ &< \|A - \text{diag } A\|_e - d_j + |a_{jj} + \lambda| < L(\lambda) \end{aligned}$$

for each $j = 1, 2, \dots$ and for all $\lambda \in \mathbb{C}$. Hence

$$\begin{aligned} \|A + C + \lambda I\| &= \sup_j \left[\sum_{i \neq j} |a_{ij} + c_{ij}| + |a_{jj} + \lambda| \right] < L(\lambda) \\ &< \|A + \lambda I\|_e \quad \text{for all } \lambda \in \mathbb{C}. \end{aligned}$$

Since $\|A + \lambda I\|_e < \|A + C + \lambda I\|$, we have $\|A + C + \lambda I\| = \|A + \lambda I\|_e$ for all $\lambda \in \mathbb{C}$.

It remains to show that C is compact. Suppose to the contrary that C is not compact. Then $\lim_{n \rightarrow \infty} \sup_j \sum_{i=1}^{\infty} |c_{ij}| = c_0 > 0$, so there are increasing sequences of positive integers $\{n_k\}$ and $\{j_k\}$, with $j_k > n_k$, such that $\lim_{k \rightarrow \infty} \sum_{i=1}^{\infty} |c_{ij_k}| = c_0$. Let λ_k be chosen so that if $d_k^* = \|A - \text{diag } A\|_e + |a_{kk} + \lambda_k| - L(\lambda_k)$, then $d_k - d_k^* < c_0/3$. By taking subsequences if necessary, we can assume that $\lim_{k \rightarrow \infty} a_{j_k j_k} = a_0$ and $\lim_{k \rightarrow \infty} d_k = d_0$. Now choose j_{k_0} so that $||a_{j_{k_0} j_{k_0}} + \lambda_{j_{k_0}}| - |d_0 + \lambda_{j_{k_0}}|| < c_0/3$ and $|d_{j_{k_0}} - d_0| < c_0/3$. Then we have

$$\begin{aligned} d_{j_{k_0}} &< d_{j_{k_0}}^* + c_0/3 = \|A - \text{diag } A\|_e + |a_{j_{k_0} j_{k_0}} + \lambda_{j_{k_0}}| - L(\lambda_{j_{k_0}}) + c_0/3 \\ &< \|A - \text{diag } A\|_e + |a_{j_{k_0} j_{k_0}} + \lambda_{j_{k_0}}| - \lim_{k \rightarrow \infty} \left[\sum_{i=1}^{\infty} |a_{ij_k}| + |a_{j_k j_k} + \lambda_{j_k}| \right] + c_0/3. \end{aligned}$$

Now since

$$\begin{aligned} \sum_{\substack{i=n_k \\ i \neq j_k}}^{\infty} |a_{ij_k}| &= \sum_{i \neq j_k} |a_{ij_k}| - \sum_{i=1}^{n_k-1} |a_{ij_k}| = \sum_{i \neq j_k} |a_{ij_k}| - \sum_{i=1}^{n_k-1} |c_{ij_k}| \\ &= \sum_{i \neq j_k} |a_{ij_k}| - \sum_{i=1}^{\infty} |c_{ij_k}| + \sum_{i=n_k}^{\infty} |c_{ij_k}| = \|A - \text{diag } A\|_e - c_{j_k} + \sum_{i=n_k}^{\infty} |c_{ij_k}| \\ &= \|A - \text{diag } A\|_e - d_{j_k} + \sum_{i=n_k}^{\infty} |c_{ij_k}|, \end{aligned}$$

we see that

$$\overline{\lim}_{k \rightarrow \infty} \left[\sum_{\substack{i=n_k \\ i \neq j_k}}^{\infty} |a_{ij_k}| + |a_{j_k j_k} + \lambda_{j_k}| \right] = \|A - \text{diag } A\|_e - d_0 + c_0 + |a_0 + \lambda_{j_{k_0}}|.$$

Hence

$$\begin{aligned} d_{j_{k_0}} &< \|A - \text{diag } A\|_e + |a_{j_{k_0} j_{k_0}} + \lambda_{j_{k_0}}| \\ &\quad - (\|A - \text{diag } A\|_e - d_0 + c_0 + |a_0 + \lambda_{j_{k_0}}|) + c_0/3 \\ &< c_0/3 + d_0 - c_0 + c_0/3 = d_0 - c_0/3 < d_{j_{k_0}}, \end{aligned}$$

which is a contradiction. Hence C is compact and the proof is complete.

THEOREM 4. *If $A \in B(l_1)$, then there exists $C \in C(l_1)$ such that $V(A + C) = V \text{ess}(A)$.*

PROOF. By [12, p. 419], we have that $p \in V(A + C)$ if and only if $|p + \lambda| < \|A + C + \lambda I\|$ for all $\lambda \in \mathbb{C}$ and $p \in V \text{ess}(A)$ if and only if $|p + \lambda| < \|A + \lambda I\|_e$ for all $\lambda \in \mathbb{C}$. Hence if C is chosen as in Theorem 3, we clearly have $V(A + C) = V \text{ess}(A)$.

REMARKS. A similar result can be obtained for those operators on l_∞ which can be represented by a matrix. In this case, we would treat the rows rather than the columns, since

$$\|A\| = \sup_i \sum_{j=1}^{\infty} |a_{ij}| \quad \text{and} \quad \|A\|_e = \lim_{n \rightarrow \infty} \sup_i \sum_{j=n}^{\infty} |a_{ij}|.$$

If $V \text{ess}(A) \subseteq R$, A is called essentially Hermitian. In [2] and [3] it is shown that Theorem 4 holds for essentially Hermitian operators on certain Orlicz sequence spaces l_M and on L_p , $1 < p < \infty$.

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