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A GEOMETRIC VARIANT OF BADE'S THEOREM ON DOMINATING MEASURES

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ABSTRACT. Let $\mathfrak{B}$ be a bounded Boolean algebra of projections in a superreflexive Banach space B. Then for each b in B there is a $\beta = \varphi(b)$ in B^* such that φ is norm-norm uniformly bicontinuous and $\beta(Pb) = 0$ if and only if $Pb = 0$.

1. Introduction. Bade (1955) showed that if B is a Banach space and if $\mathfrak{B}$ is a complete Boolean algebra of projections of B into itself, then for each b in B there is a dominating element β in B^* such that for each P in $\mathfrak{B}$, $\beta(Pb) = 0$ if and only if $Pb = 0$. From a measure-theoretic point of view, setting $\alpha(P) = \beta(Pb)$ gives a positive measure α on $\mathfrak{B}$ such that the vector measure Pb from $\mathfrak{B}$ into B is absolutely continuous with respect to α.

Bade's proof requires completeness of $\mathfrak{B}$ but asks nothing special of the space B; his proof uses the Stone representation space of $\mathfrak{B}$. The present proof uses geometric methods which require strong hypotheses on the Banach space but ask only boundedness of $\mathfrak{B}$. Also, this method gives continuous dependence of β on b.

2. Geometry of the unit ball in a Banach space. Let U and S [U^π and S'] be the unit ball and unit sphere in B [in B^*]. Define J from S to S' by: For each b in S, $Jb = \{\beta \text{ in } S' \mid \beta(b) = 1\}$. For the following facts about J see, for example, Cudia (1964).

LEMMA 2A. (i) *For each b in S, Jb is a w^*-closed convex subset of S'.*

(ii) *Jb is a singleton $\{\beta\}$ if and only if the Gateaux derivative of $\| \;\|$ exists at b and is β.*

(iii) *When the norm topology is used in S and in S', J is [uniformly] continuous on S if and only if $\| \;\|$ is [uniformly] Fréchet differentiable on S.*

LEMMA 2.1. *If $\mathfrak{B}$ is a Boolean algebra of projections in B, each of norm $\leqslant 1$, if b is in S, and if a point β in Jb exposes b on U, then $\beta(Pb) = 0$ implies that $Pb = 0$.*

PROOF. If $\beta(Pb) = 0$, then $\beta(I - P)(b) = \beta(b) = 1$. But $\|(I - P)(b)\| \leqslant 1$; that is, $(I - P)(b)$ is in $U \cap \beta^{-1}(1) = \{b\}$, because β exposes b on U. Hence $(I - P)(b) = b$ or $Pb = 0$.

COROLLARY 2.2. *If B is rotund and if $\mathfrak{B}$ is a Boolean algebra of projections in B of norm $\leqslant 1$, then each β in Jb satisfies the condition that $\beta(Pb) = 0$ implies that $Pb = 0$.*

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This is a consequence of the well-known fact that B is rotund if and only if every point of S is an exposed point of U. (See, for example, Day (1958 or 1973), VII, 2(1).)

COROLLARY 2.3. *If* $\| \ \|$ *in* B *is rotund and* [*uniformly*] *Fréchet differentiable and if* $\mathfrak{B}$ *is a Boolean algebra of projections in* B *of norm* ≤ 1, *then* J *is* [*uniformly*] *norm-norm continuous from* S *into* S' *and* $[Jb](Pb) = 0$ *if and only if* $Pb = 0$.

For proof use Lemma 2A, (ii) and (iii), and Lemma 2.1.

3. Renorming. Let $\mathfrak{B}$ be a bounded Boolean algebra of projections in B. For each P in $\mathfrak{B}$ define $V = 2P - I$. Then V is an involution and $\mathcal{V} = \{V | V = 2P - I \text{ and } P \in \mathfrak{B}\}$ is a bounded abelian group of involutions of B onto B.

For any bounded group $\mathcal{V}$ of isomorphisms of B onto B it is easy to define at least one norm on B isomorphic to the original norm and invariant under all elements of the group; let

$$p_1(b) = \sup\{\|Vb\|, V \in \mathcal{V}\}.$$

Clearly, $\|P\|_{p_1} = \|V + I\|_{p_1}/2 \leq 1$ for each P in $\mathfrak{B}$. The difficulty is that this new norm may have lost whatever roundness or smoothness properties the original norm may have had. (A picture of two smooth bathtubs intersecting in a square shows one possibility when B is two-dimensional and $\mathcal{V}$ has one reflection.) We must either renorm in a completely different way to avoid this or else renorm again to improve the norm and maintain the isometry property of each V.

If B is a complete inner-product space, a different construction (Sz. Nagy (1947); Dixmier (1950); Day (1957), p. 542) avoids the difficulty by using the amenability of each abelian group; this was proved, but not so named, by von Neumann (1929).

THEOREM 3.1. *If* H *is a complete inner-product space and if* $\mathfrak{B}$ *is a bounded Boolean algebra of projections in* H, *then there is a new isomorphic inner product* $[\ ,\ ]$ *in* H *under which each* V *in* $\mathcal{V}$ *is an isometry*, $[Px, x] = 0$ *if and only if* $Px = 0$, *and under which* $Jx = [\ , x]$ *is a uniformly continuous function of* x *on* S.

PROOF. Let $(\ ,\ )$ be the inner-product attached to the original norm in H. Let α be any invariant mean on the bounded functions on $\mathcal{V}$. For each x, y in H let $f_{x,y}(V) = (Vx, Vy)$ for all V in $\mathcal{V}$, and let $[x, y] = \alpha(f_{x,y})$. It is easily seen that this inner-product determines a $\mathcal{V}$-invariant norm isomorphic to the original norm. Also for x and y in S, $[y, x] = [Jx](y)$, so Corollary 2.3 can be used.

If B is not a Hilbert space but is superreflexive, much the same result is attainable by different methods. The next lemma is from my paper *Invariant renorming*, Day (1976). The properties of superreflexivity used can be found in Day (1973), Chapter VII, §4, B, especially, Theorem 4.

LEMMA 3A. *If* B *is superreflexive and if* G *is a group of linear isometries of* B *onto* B, *then there is a new isomorphic norm in* B *such that the new space is uniformly convex and the elements of* G *are still isometries.*

(To prove this merely follow the proof of Enflo (1972) of renormability of B, observing that each step is invariant under all elements of G.)

THEOREM 3.2. *Let B be a superreflexive space and let* $\mathfrak{B}$ *be a bounded Boolean algebra of projections in B. Then there is an isomorphic uniformly convex and uniformly Fréchet differentiable norm in B for which all P in* $\mathfrak{B}$ *are of norm* < 1, *J is uniformly continuous, as is* J^{-1}, *and* $[Jb](Pb) = 0$ *if and only if* $Pb = 0$, *and* $[P^*\beta](J^{-1}\beta) = 0$ *if and only if* $P^*\beta = 0$.

PROOF. As before, construct $\mathcal{V} = \{V = 2P - I \mid P \in \mathfrak{B}\}$. Then construct the $\mathcal{V}$-invariant norm p_1, defined in line 8 of this section. The space with norm p_1 is still superreflexive but the elements of $\mathcal{V}$ are now isometries. Use Lemma 3A with $\mathcal{V}$ and p_1 to get a new uniformly convex norm p_2 isomorphic to p_1 and still invariant under $\mathcal{V}$. Use Lemma 3A again with $\mathcal{V}^*$ and p_2^* in the superreflexive conjugate space B^* to get a uniformly convex norm p_3^* in B^*; then p_3^* is conjugate to some norm p_3 in B and p_3 will also be $\mathcal{V}$-invariant. Construct the Asplund (1967) average of p_2 and p_3 and call it p_4. Then p_4 will be both uniformly rotund and uniformly smooth, and will also be invariant under $\mathcal{V}$. By Corollary 2.3, J is uniformly continuous and $[Jb](Pb) = 0$ if and only if $Pb = 0$. Since J^{-1} is just the corresponding map defined from B^* onto B, J^{-1} is also uniformly continuous, and $[P^*\beta](J^{-1}\beta) = 0$ if and only if $P^*\beta = 0$.

COROLLARY 3.3. *If* $\mathfrak{B}$ *is a bounded Boolean algebra in a superreflexive space B, then there is a biuniformly continuous homeomorphism* Φ *of S onto S′ such that for each P in B,* $[\Phi b](Pb) = 0$ *if and only if* $Pb = 0$, *and also* $[P^*\beta](\Phi^{-1}\beta) = 0$ *if and only if* $P^*\beta = 0$.

PROOF. Temporarily, let S and Σ [S' and Σ'] be unit spheres for $\|\ \|$ and for p_4 [for $\|\ \|^*$ and for $p_4{}^*$]. Define the radial maps $\rho(b) = b/p_4(b)$ from S onto Σ, and $r(\beta) = \beta/\|\beta\|^*$ from Σ' onto S'. These functions are biuniformly continuous (in fact, Lipschitz) homeomorphisms. If $\Phi(b) = r(J(\rho(b)))$, then Φ is uniformly continuous from S onto S', and $\Phi^{-1} = \rho^{-1}J^{-1}r^{-1}$ is also uniformly continuous from S' onto S. Thus Φ also has the other properties of the corollary.

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