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A COUNTEREXAMPLE FOR COMMUTATION IN TENSOR PRODUCTS OF C^*-ALGEBRAS

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ABSTRACT. An example is given to show the failure of the analogue for C^*-algebras of the commutation theorem for von Neumann tensor products.

Let A, B, C, D be C^*-algebras with $A \subseteq C$ and $B \subseteq D$. Tomiyama [**10**, p. 29] has raised the question as to whether $(A \otimes B)^c = A^c \otimes B^c$. In this context $\otimes$ denotes the spatial tensor product and $(A \otimes B)^c$ (respectively A^c, B^c) is the relative commutant of $A \otimes B$ (respectively A, B) in $C \otimes D$ (respectively C, D). It is easy to see that the inclusion $A^c \otimes B^c \subseteq (A \otimes B)^c$ is always valid. In the special case where C has an identity 1, $A = \mathbf{C}1$ and $B = D$, the question has an affirmative answer [**7**, Theorem 1], and the result has been generalized to the case of an arbitrary C^*-tensor norm [**1**], [**4**]. It is therefore tempting to conjecture that the question has an affirmative answer at least in the case where $A = \mathbf{C}1$ (but B is an arbitrary C^*-subalgebra of D). In this note we present a counterexample based on results of Choi [**5**], Wassermann [**13**] and Voiculescu [**11**].

We begin by recalling some facts about slice maps [**9**]. Let A and B be C^*-algebras and let $\phi \in A^*$. The *right slice map* R_ϕ: $A \otimes B \to B$ is the unique bounded linear mapping with the property that $R_\phi(a \otimes b) = \phi(a)b$ ($a \in A$, $b \in B$). A triple (A, B, J), where J is a closed two-sided ideal of B, is said to *verify the slice map conjecture* [**12**] if whenever $x \in A \otimes B$ and $R_\phi(x) \in J$ for all $\phi \in A^*$ then $x \in A \otimes J$. It is well known that (A, B, J) verifies the slice map conjecture if and only if $A \otimes J$ is the kernel of the canonical $*$-homomorphism μ: $A \otimes B \to A \otimes (B/J)$. This is because

$$\ker \mu = \{x \in A \otimes B \,|\, R_\phi(x) \in J \text{ for all } \phi \in A^*\}.$$

The following result is implicit in [**13**, 2.5]. Although we shall apply it in a rather special case, we state it in the given form since it may be of independent interest.

PROPOSITION. *Suppose that B/J is a nuclear C^*-algebra. Then the triple (A, B, J) verifies the slice map conjecture.*

PROOF. Suppose that (A, B, J) does not verify the slice map conjecture. The canonical $*$-isomorphism of the algebraic tensor product $A \odot (B/J)$ into $(A \otimes B)/(A \otimes J)$ induces on $A \odot (B/J)$ a C^*-norm which is distinct from the least C^*-norm since $\ker \mu \neq A \otimes J$. This contradicts the nuclearity of B/J.

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REMARK. Since any quotient of a nuclear C^*-algebra is nuclear [6], it follows from the Proposition that (A, B, J) verifies the slice map conjecture whenever B is a nuclear C^*-algebra. It is shown in [2] that in fact it suffices to assume that B is just a C^*-subalgebra of some nuclear C^*-algebra in order to conclude that (A, B, J) verifies the slice map conjecture.

We now give the counterexample. Let $C = C^*(F_2)$, the full C^*-algebra of the free group on two generators, and let J be the kernel of the canonical $*$-homomorphism from C onto the C^*-algebra of the left regular representation of F_2. Since C is separable it follows from [8, 3.7.5] that there is a faithful (nondegenerate) representation π of C on a separable Hilbert space H (of infinite dimension). By [5, Corollary 2] $\pi(C)$ contains no nonzero compact operator. It follows that we may regard C as a C^*-subalgebra of the Calkin algebra (D say) associated with H. Since π was nondegenerate we may assume that the identity 1 of D lies in C. Let $A = \mathbf{C}1$ and let $B = (J + \mathbf{C}1)^c$ (the relative commutant of $J + \mathbf{C}1$ in D). Then, with the relative commutants taken as indicated in the opening paragraph, we have the following result.

THEOREM. $(A \otimes B)^c \neq A^c \otimes B^c$.

PROOF. Since $J + \mathbf{C}1$ is a separable C^*-subalgebra of D it follows from [11] (see also [3, p. 345]) that $(J + \mathbf{C}1)^{cc} = J + \mathbf{C}1$. Thus $B^c = J + \mathbf{C}1$ and so $A^c \otimes B^c = C \otimes (J + \mathbf{C}1)$. By [13, 2.7, Remark] there exists $x \in C \otimes C$ ($\subseteq C \otimes D$) such that $x \notin C \otimes J$ and $R_\phi(x) \in J$ for all $\phi \in C^*$ (where R_ϕ is the right slice map $C \otimes C \to C$ associated with ϕ). Applying the Proposition to the triple $(C, J + \mathbf{C}1, J)$ we see that $x \notin C \otimes (J + \mathbf{C}1)$.

Let $b \in B$. For $\phi \in C^*$ we have

$$R_\phi[x(1 \otimes b) - (1 \otimes b)x] = R_\phi(x)b - bR_\phi(x) = 0.$$

Hence $x(1 \otimes b) - (1 \otimes b)x = 0$ [9, Theorem 1], and so $x \in (A \otimes B)^c$.

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