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LEVI FLAT HYPERSURFACES WHICH ARE NOT HOLOMORPHICALLY FLAT

ERIC BEDFORD AND PAOLO DE BARTOLOMEIS

ABSTRACT. A real analytic, Levi flat hypersurface $S \subset \mathbf{C}^n$ is locally biholomorphically flat. It is shown here that if S is Levi flat and C^∞, then in general it is not possible to flatten S, even in a local, "one-sided" sense.

If $S \subset \mathbf{C}^{n+1}$ is a smooth hypersurface whose Levi form vanishes identically then S is a 1-parameter family of complex manifolds of dimension n. If S is real analytic, then locally one may choose holomorphic coordinates z^* such that $S = \{\operatorname{Re} z^*_{n+1} = 0\}$. Let us fix $p \in S$ and fix one side of S at p; we let D^+ denote a small one-sided neighborhood of S at p, which, after shrinking, will be again denoted by D^+.

We consider the possibility of a local one-sided flattening of S in the following sense.

$$\text{There exists a holomorphic map } F\colon D^+ \to \{z \in \mathbf{C}^{n+1}: \operatorname{Re} z_{n+1} > 0\} \text{ such that if } \{\zeta_j\} \subset D^+ \text{ and } \lim_{j\to\infty} \operatorname{dist}(\zeta_j, S) = 0 \text{ then } \lim_{j\to\infty} \operatorname{dist}(F(\zeta_j), \{\operatorname{Re} z_{n+1} = 0\}) = 0. \tag{1}$$

It is easily seen that S cannot be flattened in the sense of (1) if the Levi form of S does not vanish identically (cf. condition (c) of the lemma). Related results have been obtained by Henkin [4] and Pinčuk [5]. The point of this paper is that Levi flatness is not sufficient for the surface S to be flattened.

The complex n-manifolds in S form a C^∞ foliation $\mathfrak{F}$ of S. A related problem is to ask whether $\mathfrak{F}$ can be extended to a holomorphic foliation of D^+. (Recall that a foliation of codimension 1 is *holomorphic* if locally there are coordinates z^* such that the leaves are given as $\{z^*_{n+1} = c\}$.)

LEMMA. *The following are equivalent near a point p of a smooth Levi flat hypersurface $S \subset \mathbf{C}^{n+1}$.*

(a) *S can be flattened in the sense of* (1),

(b) *S can be flattened in the sense of* (1), *and we may take the holomorphic mapping F in* (1) *to be smooth on $\overline{D}^+$ and $F\colon D^+ \to F(D^+)$ is to be a biholomorphism,*

(c) *there is a pluriharmonic function $h \in C^\infty(\overline{D}^+)$ such that $h > 0$ on D^+ and $h = 0$ on $S \cap \overline{D}^+$, and*

(d) *there is a C^∞ foliation $\tilde{\mathfrak{F}}$ of $D^+ \cup (S \cap \overline{D}^+)$ which is holomorphically trivial on D^+ and such that $\tilde{\mathfrak{F}} = \mathfrak{F}$ on $S \cap \overline{D}^+$.*

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PROOF. (a) $\Rightarrow$ (b) If F is the mapping given by (1), then $\operatorname{Re} F_{n+1}$ is pluriharmonic on D^+ and $\lim_{\zeta \in D^+, \zeta \to S} \operatorname{Re} F_{n+1} = 0$. Thus

$$\operatorname{Re} F_{n+1} \in C^\infty(D^+ \cup (S \cap \bar{D}^+)).$$

It follows, then, that F_{n+1} is also C^∞. Now we may assume that $p = 0$, and that $\partial/\partial z_j$ is tangent to S at 0 for $1 \leq j \leq n$. Thus we may replace the original mapping F by $(z_1, \ldots, z_n, F_{n+1}(z))$. By the Hopf lemma $\partial F_{n+1}/\partial z_{n+1} \neq 0$, so this is a local biholomorphism.

(b) $\Rightarrow$ (c) This holds because we may take $h = \operatorname{Re} F_{n+1}$.

(c) $\Rightarrow$ (d) If we let $\tilde{h}$ denote the pluriharmonic conjugate of h, then $H = h + i\tilde{h} \in \mathcal{O}(D) \cap C^\infty(D^+ \cup (S \cap \bar{D}^+))$. We define the leaves of $\tilde{\mathfrak{F}}$ as the level sets $\{z \in \bar{D}^+ : H(z) = c\}$. By the Hopf lemma, $\partial h \neq 0$ on S, so the leaves are smooth, and $\tilde{\mathfrak{F}}$ is holomorphic on D^+. Since the leaves of $\mathfrak{F}$ are uniquely determined by integrating ∂h on S, it follows that $\tilde{\mathfrak{F}}$ extends $\mathfrak{F}$.

(d) $\Rightarrow$ (a) If $d \operatorname{Im} z_{n+1}$ is the normal covector to S at $p = 0$, then we let $\Gamma^+ = \{z \in D^+ : z_1 = \cdots = z_n = 0\}$. Shrinking D^+ suitably, we may find a conformal equivalence $f: \Gamma^+ \to \{\zeta \in \mathbf{C}: \operatorname{Re} \zeta > 0\}$ such that $f(0) = 0$ and $\operatorname{Re} f = 0$ on $\bar{\Gamma}^+ \cap S$.

We may also assume that each leaf M of $\tilde{\mathfrak{F}}$ is closed and intersects $\Gamma^+ \cup (S \cap \bar{\Gamma}^+)$ in exactly one point. Now we define $\tilde{f}$ on $D^+ \cup (S \cap \bar{D}^+)$ by making it constant on the leaves of $\tilde{\mathfrak{F}}$ and setting $\tilde{f}|_{\Gamma^+ \cup (S \cap \bar{\Gamma}^+)} = f$. Clearly $\tilde{f} \in C^\infty(D^+ \cup (S \cap \bar{D}^+))$, and $\tilde{f} \in \mathcal{O}(D^+)$ since $\tilde{\mathfrak{F}}$ is holomorphic. Now $F = (z_1, \ldots, z_n, \tilde{f})$ gives the desired map in (a).

C. Rea [6] has given an example of a C^∞ surface S such that the foliation $\mathfrak{F}$ does not have an extension to be holomorphic in a two-sided neighborhood of p.

Let us assume that $\operatorname{Im} dz_{n+1}$ is normal to S at $p = 0$ and that $\operatorname{Im} dz_{n+1}$ points toward D^+. We may parametrize the foliation $\mathfrak{F}$ of S by $G: \Delta^n \times I \to \mathbf{C}^{n+1}$, $I = [-1, 1] \subset \mathbf{R}$, $\Delta = \{\zeta \in \mathbf{C}: |\zeta| < 1\}$.

$$G(z, t) = (z, \varphi(t) + g(z, t)), \tag{2}$$

$\varphi(t) \in C^\infty(I)$, $\varphi' \neq 0$, $\varphi'(0) > 0$, $g(z, t) \in C^\infty(\Delta^n \times I)$, $g(0, t) = 0$, and g is holomorphic in z for fixed t.

Locally, then, we have $S = G(\Delta^n \times I)$. It will also be useful to parametrize S slightly differently.

Let $\gamma = \varphi(]-\varepsilon, \varepsilon[)$ be the image of $]-\varepsilon, \varepsilon[$ under φ, and let Γ be a small neighborhood of γ in $\mathbf{C}$ such that $\Gamma = \Gamma^+ \cup \gamma \cup \Gamma^-$, where $\Gamma^\pm$ are connected open sets. We may also write $S = G(\Delta^n \times \gamma)$ where we set

$$G(z, \eta) = (z, \eta + g(z, \varphi^{-1}(\eta))) \tag{3}$$

for $z \in \Delta^n$, $\eta \in \gamma$. Since the graph of $g(z, \eta_0)$ for a fixed $\eta_0 \in \gamma$ is the (unique) leaf of $\mathfrak{F}$ passing through $(0, \eta_0)$ we see that S uniquely determines $g(z, \eta)$ in (3).

For functions $\varphi, a \in \mathbf{C}^\infty(]-\varepsilon, \varepsilon[)$, we will use the following extendibility criterion.

There is an analytic function $A(\eta) \in \mathcal{O}(\Gamma^+) \cap C^\infty(\Gamma^+ \cup \gamma)$ such that $A(\eta) = a(\varphi^{-1}(\eta))$ for all $\eta \in \gamma$. (4)

REMARK. We note that (4) is a nontrivial criterion which is not satisfied for general φ, $a \in C^\infty(]-\varepsilon, \varepsilon[)$. For example, we may take $\varphi(t) = t$ and let $a^+, a^- \in C^\infty(\mathbf{R})$ be functions that are nowhere real analytic but such that $a^\pm(t)$ extends to be holomorphic in $\{\operatorname{Im} \zeta \gtrless 0\}$. Then the pair $(\varphi, a^+ + a^-)$ does not satisfy (4) on any interval.

THEOREM. *Let the surface S be a C^∞, Levi flat hypersurface in $\mathbf{C}^{n+1}$, and let us represent S locally in the form* (3). *Then S can be locally flattened at $0 \in S$ in the sense of* (1) *if and only if the pair $(\varphi(t), g(z, t))$ satisfies* (4) *holomorphically in z, i.e. there is a function $\tilde{g} \in \mathcal{O}(\Delta^n \times \Gamma^+) \cap C^\infty(\Delta^n \times (\Gamma^+ \cup \gamma))$ such that $\tilde{g}|_{\Delta^n \times \gamma} = g(z, \varphi^{-1}(\eta))$.*

Before giving the proof we state a corollary.

COROLLARY. *There is a C^∞, Levi flat hypersurface S that cannot be mapped* (*in the local one-sided sense of* (1)) *to a real analytic Levi flat surface S' in such a way that $\lim_{\zeta \to p, \zeta \in D^+} F(\zeta) = p' \in S'$ exists.*

The corollary follows from the Theorem since we may biholomorphically flatten S' in a neighborhood of p' and we may choose S to be given by a pair (φ, g) which does not satisfy (4).

This is analogous to a result of Faran [2] for strongly pseudoconvex domains. Faran's result uses the local invariants of Chern and Moser [1] and the boundary regularity of F, proved by Fefferman [3]. In our case, there are neither local invariants nor boundary regularity for biholomorphisms.

PROOF OF THE THEOREM. First we show that S can be flattened if $\tilde{g}(z, \eta)$ is given. We note that $dg(0, t)/dt \neq 0$, and thus $\tilde{F}$: $\Delta^n \times \Gamma^+ \to D^+$ given by $\tilde{F}(z, \eta) = (z, \eta + \tilde{g}(z, \eta))$ is a biholomorphism of a one-sided neighborhood of 0. By the Riemann mapping theorem there is a holomorphic equivalence

$$\psi: \Gamma^+ \to \{\zeta \in \Delta: \operatorname{Im} \zeta > 0\}$$

such that $\psi(0) = 0$. Restricting to a small neighborhood of 0, we see that ψ maps γ to the real axis. If $\Psi(z, \eta) = (z, \psi(\eta))$ then $\Psi(\tilde{F}^{-1}) = F$ is the desired mapping for (1).

Conversely, we suppose that S can be flattened and show that $(\varphi(t), g(z, t))$ satisfy (4).

By the Lemma, there is a pluriharmonic $h \in C^\infty(D^+ \cup (S \cap \bar{D}^+))$ such that $h = 0$ on S and $h > 0$ on D^+. Let $\tilde{h}$ be a pluriharmonic conjugate so that $H = h + i\tilde{h} \in \mathcal{O}(D^+)$. We define $\tilde{g}(z, \eta)$ on $\Delta^n \times \Gamma^+$ by

$$H(z, \eta + \tilde{g}(z, \eta)) = H(0, \eta) \tag{5}$$

for all $z \in (\Delta')^n$ where $\Delta' \subset\subset \Delta$ is some smaller disk. Since $\partial H/\partial \eta \neq 0$ on $\Delta \times \{0\}$, we see that we may use (5) to define $\tilde{g}$: $\Delta^n \times \Gamma^+ \to \mathbf{C}$ implicitly (recall that we are free to shrink Γ^+). Since $H(0, \eta)$ is analytic in η for $\eta \in \Gamma^+$ we see that $\tilde{g} \in \mathcal{O}(\Delta^n \times \Gamma^+) \cap C^\infty(\Delta^n \times (\Gamma^+ \cup \gamma))$. Further, since for fixed $\eta_0 \in \gamma$ the graph of $\tilde{g}(z, \eta_0)$ is the leaf of $\mathfrak{F}$ passing through $(0, \eta_0)$, it follows that $\tilde{g}(z, \eta_0) = g(z, \varphi^{-1}(\eta_0))$ and thus (φ, g) satisfies (4).

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Department of Mathematics, Princeton University, Princeton, New Jersey 08544 (Current address of Eric Bedford)

School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540

Current address (Paolo de Bartolomeis): Istituto Matematico "U. Dini", viale Morgagni 67/A, I-50134 Firenze, Italy