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A CHARACTERIZATION OF TOTALLY GEODESIC HYPERSURFACES OF S^{n+1} AND $\mathbf{C}P^{n+1}$

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ABSTRACT. Let M be a complete hypersurface in S^{n+1} (or $\mathbf{C}P^{n+1}$). Assume that through each point x of M a (local) $\mu(x)$-dimensional totally geodesic submanifold S_x of S^{n+1} (or $\mathbf{C}P^{n+1}$) exists in M. A sufficient condition for M itself to be totally geodesic is given in terms of $\mu(x)$.

Introduction. In this paper, we give a sufficient condition for a hypersurface of S^{n+1} (or $\mathbf{C}P^{n+1}$ in the complex case) to be totally geodesic.

Obviously, if a hypersurface M of S^{n+1} (or $\mathbf{C}P^{n+1}$) is locally totally geodesic everywhere in M, then M is globally totally geodesic. We ask, therefore, what we can say about M if M contains a lower dimensional totally geodesic submanifold (local) through each point of M. The specific statement of this assumption is given as condition (*) (or (**) for the complex case) in the following sections.

Our main results are stated as Theorems A and A′. These theorems basically tell us that we can often conclude that M is totally geodesic, even if the dimensions of the local totally geodesic submanifolds are considerably lower than that of M itself. In particular, the dimensions of local totally geodesic submanifolds in the complex case may be as low as a half of that of $M + 1$.

Our condition (*) (or (**)) may also be regarded as an extremely simplified version of the so-called axiom of spheres for hypersurfaces of S^{n+1} (or $\mathbf{C}P^{n+1}$); see Cartan [5] or Leung-Nomizu [6] for this notion.

Hypersurfaces of S^{n+1}. Let us denote by S^{n+1} the standard $(n+1)$-dimensional sphere of constant curvature c. Let M be a complete Riemannian manifold of dimension n. Furthermore, assume that there is an isometric immersion f of M into S^{n+1}. Call the pair (M, f) a hypersurface of S^{n+1}. Our condition (*) is stated as follows.

(*) Through each point x of M exists a $k(x)$-dimensional submanifold S_x $(2k(x) > n)$ of M which is mapped under f isometrically into a $k(x)$-dimensional totally geodesic sphere of S^{n+1}. Here $k(x)$ is a positive integer-valued function of M.

Let (M, f) be as above. Denote by $\overline{\nabla}$ and ∇ the Riemannian connections of S^{n+1} and M, respectively. Then, for any tangent vector fields X and Y of M, we have

$$\overline{\nabla}_{f_*(X)} f_*(Y) = f_*(\nabla_X Y) + \alpha(X, Y),$$

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where α is the second fundamental form of (M, f). The relative nullspace RN_x at $x \in M$ is defined by $RN_x = \{X \in TM_x : \alpha_x(X, Y) = 0$ for all $Y \in TM_x\}$, where TM_x denotes the tangent space of M at x. Let ξ be a unit normal field of M. We have the so-called shape operator A_ξ at each $x \in M$; for any $X \in TM_x$, $\overline{\nabla}_{f_*(x)}\xi = -A_\xi X + D_x\xi$, where D is the normal connection. Then, RN_x can be identified with the nullspace of A_ξ at x. The dimension $\nu(x)$ of RN_x is called the relative nullity of (M, f) at x and the minimum ν of $\nu(x)$ in M is called the index of relative nullity of (M, f).

Lemma A. *Let (M, f) be a hypersurface of S^{n+1} with $(*)$. If $2k(x) > n$ everywhere, the relative nullity $\nu(x)$ of (M, f) is greater than or equal to $2k(x) - n$. In particular, $\nu \geqslant 2k - n$, where $k = \min k(x)$, $x \in M$.*

Proof. Let x be a point in M. By condition $(*)$, S_x is locally totally geodesic in M. Let $e_1, \ldots, e_k$ form a local orthonormal frame field of S_x. Denote by $\{e_{k+1}, \ldots, e_n\}$ a local orthonormal frame field of the normal bundle of S_x in M. Clearly, $e_1, \ldots, e_k, e_{k+1}, \ldots, e_n$ form a local frame field of TM, the tangent bundle of M. Then for any e_i and e_j $(1 \leqslant i, j \leqslant n)$,

$$\overline{\nabla}_{f_*(e_i)} f_*(e_j) = f_*(\nabla_{e_i} e_j) + \alpha(e_i, e_j).$$

In particular, $\alpha(e_i, e_j) = 0$ for $1 \leqslant i, j \leqslant k$. Thus, using the shape operator A_ξ, we have the following matrix representation of A_ξ with respect to the ordered basis $e_1, \ldots, e_n$ for TM_x;

$$A_\xi = \begin{bmatrix} 0 & B \\ B^t & C \end{bmatrix}.$$

Here 0 is the $k \times k$-zero matrix, B is a $k \times (n-k)$-matrix, B^t is the transpose of B and C is an $(n-k) \times (n-k)$-symmetric matrix. Let RB be the row-reduced echelon matrix of B. Then it is clear that at least $k - (n-k) = 2k - n$ rows from below in RB must consist of the zero components. Denote the number of the zero rows of RB by $r(x)$. Hence, a $k \times n$-matrix $[0, RB]$ has the same number of zero rows at the bottom of the matrix. Considering the above sequence of row operations performed on B to get RB as a sequence of column operations applied to B^t, we get the column-reduced echelon matrix CB^t of B^t. Note here that $CB^t = (RB)^t$. Therefore, CB, and consequently the $n \times k$-matrix $[{}^{0}_{CB^t}] = [0, RB]^t$, has at least $r(x)$ zero columns counted from the right-hand side of the matrix. Now after appropriate numbers of interchanging between rows and between columns of A_ξ, we may get an $n \times n$-matrix whose first $r(x)$ rows and columns consist entirely of zeros. Since $\nu(x)$ is the multiplicity of zero as an eigenvalue of A_ξ and since the multiplicity of zero as an eigenvalue of A_ξ is invariant under the row and column operations, we get that $\nu(x) \geqslant r(x) \geqslant 2k(x) - n$. This completes the proof of Lemma A.

Let G be the open subset of M given by $G = \{x \in M: \nu(x) = \nu\}$. On G is defined the so-called relative nullity distribution RN by assigning RN_x to each x. It is well known that RN is differentiable and involutive in G. Moreover, the leaves of

RN are complete and totally geodesic in M as well as in S^{n+1}, i.e., the leaves are isometric to some ν-dimensional great spheres of S^{n+1}.

Let $s(l)$ be the largest integer such that the standard fibration $V_{e,s(e)} \to V_{e,l}$ of stiefel manifolds has a global cross-section, where $V_{a,b}$ denotes the set of all ordered b-planes in an a-dimensional Euclidean space $\mathbf{R}^a$. For any integer $n > 0$, set μ_n to be the largest integer such that $s(n - \mu_n) \geqslant \mu_n + 1$. μ_n is actually given by $n = 2^i + \mu_n$, where i is the largest power of 2 such that $2^i < n$. For example, $\mu_1 = 0$, $\mu_2 = 0$, $\mu_3 = 1$, $\mu_4 = 0$, $\mu_5 = 1$, $\mu_6 = 2$, $\mu_7 = 3$, $\mu_8 = 0, \ldots,$ etc. Using a result of Ferus [4], we obtain the following. The proof is almost exactly the same as the one given in [1].

LEMMA B. *Let (M, f) be given as above in $(*)$. If $\nu > \mu_{n+1}$, then $\nu = n$ and (M, f) is a totally geodesic imbedding of M into S^{n+1}; therefore, $M = S^n$.*

Combining Lemmas A and B, we get

THEOREM A. *Let (M, f) be a complete hypersurface of S^{n+1} with condition $(*)$. If $2k - n > \mu_{n+1}$, where $k =$ the minimum of $k(x)$ in M, then $M^n = S^n$ and f is an imbedding of S^n into S^{n+1} as a great sphere.*

Recently we showed in [2],

LEMMA C. *Let (M^n, f) be a complete n-dimensional Riemannian submanifold of S $(N > n)$. If the Ricci curvatures of M^n are greater than or equal to $(n - 1)c$ everywhere, then ν equals either 0 or n.*

Using Lemma C, we get

THEOREM B. *Let (M, f) be a complete Riemannian hypersurface of S^{n+1} with condition $(*)$. Furthermore, we assume that the Ricci curvatures of M are greater than or equal to $(n - 1)c$ everywhere. Then if $2k - n > 0$, $M^n = S^n$ and f imbeds M as a great sphere of S^{n+1}.*

REMARK. If we assume that S_x is complete for all $x \in M$, then by a result of Ferus [4], we may weaken condition $(*)$ by simply assuming that S_x is a manifold with constant curvature c instead of requiring that S_x is totally geodesic in S^{n+1}.

Holomorphic hypersurfaces in $\mathbf{C}P^{n+1}$. Let M be a complete complex n-dimensional Kählerian manifold and let $\mathbf{C}P^{n+1}$ be the complex $(n + 1)$-dimensional projective space of constant holomorphic sectional curvature c. Let $f: M \to \mathbf{C}P^{n+1}$ be a holomorphic and isometric immersion of M into $\mathbf{C}P^{n+1}$. We further assume the following condition $(**)$.

$(**)$ Through each point $x \in M$ exists a complex space form (local) S_x of complex dimension $k(x)$ $(2k(x) - n > 0)$ with constant holomophic sectional curvature c.

Note that $(**)$ is weaker than the corresponding $(*)$ in the real case, because it is not assumed that S_x is totally geodesic in $\mathbf{C}P^{n+1}$.

Let $e_1, Je_1, \ldots, e_k, Je_k, e_{k+1}, Je_{k+1}, \ldots, e_n, Je_n$ be a unitary frame field of TM in a neighborhood of x such that $e_1, Je_1, \ldots, e_k, Je_k$ are tangential to S_x. Let ξ and

$J\xi$ be a unitary normal field in the neighborhood of x. If we denote the shape operators with respect to ξ and $J\xi$ by A_ξ and $A_{J\xi}$, we have $A_{J\xi} = JA_\xi = -A_\xi J$. As in the real case, the matrix representation of A_ξ with respect to the ordered basis

$$(e_1, Je_1, \ldots, e_k, Je_k, e_{k+1}, Je_{k+1}, \ldots, e_n, Je_n)$$

has the following form:

$$A_\xi = \begin{bmatrix} 0 & B \\ B^t & C \end{bmatrix},$$

where 0 is the $2k \times 2k$-zero matrix, B is a $2k \times 2(n-k)$-matrix, B^t is the transpose of B and C is a $2(n-k) \times 2(n-k)$-symmetric matrix. Then a proof identical to that of Lemma A shows us that the nullity of A_ξ is greater than or equal to $2(2k-n) > 0$. Since $A_\xi J = -JA_\xi$, the null vectors of A_ξ and $A_{J\xi}$ coincide. It is also well known that the relative null space of (M, f) is a complex space, i.e., it is invariant under J. Thus, if we call the complex dimension of the relative null space at x the complex relative nullity $\nu(x)$, we have

LEMMA A′. *Let (M, f) be a Kählerian hypersurface of $\mathbf{C}P^{n+1}$ with condition* (**). *Then $\nu(x) \geqslant 2k - n$.*

In [1], we showed,

LEMMA B′. *Let (M^n, f) be a Kählerian submanifold of $\mathbf{C}P^N$ $(N > n)$. Then the complex index of relative nullity equals either 0 or $n = \dim M^n$. If $\nu = n$, then $M^n = \mathbf{C}P^n$ and f is a totally geodesic imbedding of $\mathbf{C}P^n$ into $\mathbf{C}P^N$.*

Combining Lemmas A′ and B′, we get

THEOREM A′. *Let (M, f) be a complete Kählerian hypersurface of $\mathbf{C}P^{n+1}$ with condition* (**). *If $2k - n > 0$, then $M = \mathbf{C}P^n$ and f is a totally geodesic imbedding of $\mathbf{C}P^n$ into $\mathbf{C}P^{n+1}$.*

BIBLIOGRAPHY

1. K. Abe, *Application of a Riccati type differential equation to Riemannian manifolds with totally geodesic distributions*, Tôhoku Math. J. **25** (1973), 425–444.

2. ______, *Some remarks on a class of submanifolds in space forms of non-negative curvature*, Math. Ann. **247** (1980), 275–280.

3. D. Ferus, *Totally geodesic foliation*, Math. Ann. **188** (1970), 313–316.

4. ______, *Isometric immersions of constant curvature manifolds*, Math. Ann. **217** (1975), 155–156.

5. E. Cartan, *Leçons sur la géometrie des espaces de Riemann*, Gauthier-Villars, Paris, 1946.

6. D. S. Leung and K. Nomizu, *The axiom of sphere in Riemannian geometry*, J. Differential Geometry **5** (1971), 487–489.

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