QUADRATIC INTEGRATION OF GAUSSIAN PROCESSES

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ABSTRACT. Let $x(t)$, $0 \leq t \leq T$, be a Gaussian process whose covariance function $R(s, t)$ satisfies certain conditions. If $G(x)$ satisfies some mild condition, then the quadratic integral $L^2\text{-}\lim \sum_k G(x(t_k))\Delta x(t_k)^2$ along any sequence of partitions of $[0, T]$ whose mesh goes to zero exists. The differential rule for $x(t)$ is also derived.

0. Introduction. Let $x(t)$, $0 \leq t \leq 1$, be a Gaussian process with covariance function $R(s, t)$, $0 \leq s, t \leq 1$. In studying some connections between Gaussian processes and boundary value problems, Baxter showed, under certain conditions, (see [1, Theorem 1, p. 522], or [6, Theorem 20.4, p. 297]) that the quadratic variation of $x(t)$ over $[0, 1]$ is

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{2^n-1} \Delta x(t_k)^2 = \int_0^1 f(t) dt \quad (0.1)$$

where $t_k = t_k^n = k2^{-n}$, $0 \leq k \leq 2^n$, $\Delta x(t_k) = x(t_{k+1}) - x(t_k)$, $0 \leq k \leq 2^n - 1$, and where $f(t)$, $0 \leq t \leq 1$, is defined by

$$f(t) = \lim_{h \uparrow 0} \{R(t+h, t) - R(t, t)\}/h - \lim_{h \downarrow 0} \{R(t+h, t) - R(t, t)\}/h. \quad (0.2)$$

This result was later extended to a wider class of processes (see [5], [2], [4]). The limit in (0.1) is L^2 -convergent (as well as almost sure convergent). If $G(y)$ is a nice function of y , it is not hard to prove that

$$\lim_{n \rightarrow \infty} E \left\{ \sum_{k=0}^{2^n-1} G(x(t_k))\Delta x(t_k)^2 \right\} = E \left\{ \int_0^1 G(x(t))f(t) dt \right\}. \quad (0.3)$$

From (0.1) and (0.3), it is reasonable to conjecture that the quadratic integral of $G(x(t))$ w.r.t. $x(t)$ is

$$L^2\text{-}\lim_{n \rightarrow \infty} \sum_{k=0}^{2^n-1} G(x(t_k))\Delta x(t_k)^2 = \int_0^1 G(x(t))f(t) dt. \quad (0.4)$$

We shall prove that this is really the case under some suitable conditions.

Some basic properties of normal distributions are listed in §1. The proof of (0.4) is given in §2. As an application, Ito's lemma for $x(t)$ is derived in the last section.

1. Basic properties. In this section, we list some properties of Gaussian distribution which will be needed in the next section. Since the proofs of these properties are routine, they are omitted.

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PROPERTY 1.1. Let X be a normal random variable with mean 0 and variance σ^2 . Then for each $m > 0$, there exists a constant c_m such that $E\{(X^2 - \sigma^2)^m\} = c_m \sigma^m$.

PROPERTY 1.2. Let Y_1, Y_2 have a bivariate normal distribution with means μ_1, μ_2 and variances σ_1^2, σ_2^2 respectively. Let ρ denote their covariance. Then

$$EY_1^2Y_2^2 = \sigma_1^2\sigma_2^2 + 2\rho^2 + \mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2 + 4\mu_1\mu_2\rho + \mu_1^2\mu_2^2.$$

Let random variables $Z_i, 1 \leq i \leq 4$, have a multivariate normal distribution with zero means. Let $B = (b_{ij})$ denote their covariance matrix and let B_{ij} denote the cofactor of b_{ij} for each i, j . Let $C = (c_{ij})$ denote the covariance matrix of Z_1, Z_2, Z_3 and let C_{ij} denote the cofactor of c_{ij} . Then we have

PROPERTY 1.3. The conditional density $N(z_1, z_2|z_3, z_4)$ of Z_1, Z_2 given $Z_3 = z_3, Z_4 = z_4$ is bivariate normal. Let $m_1, m_2; V_1, V_2; \Gamma$ denote the conditional means, variances and covariance respectively. Then

$$\begin{aligned} m_1 &= \{(B_{12}B_{23} - B_{13}B_{22})z_3 + (B_{12}B_{24} - B_{14}B_{22})z_4\} / (B_{11}B_{22} - B_{12}^2), \\ m_2 &= \{(B_{12}B_{13} - B_{11}B_{23})z_3 + (B_{12}B_{14} - B_{11}B_{24})z_4\} / (B_{11}B_{22} - B_{12}^2), \\ V_1 &= B_{22}|B| / (B_{11}B_{22} - B_{12}^2), \quad V_2 = B_{11}|B| / (B_{11}B_{22} - B_{12}^2), \\ \Gamma &= B_{12}|B| / (B_{11}B_{22} - B_{12}^2) \end{aligned} \quad (1.1)$$

where $|B|$ denotes the determinant of B .

PROPERTY 1.4. The conditional density $N(z_1, z_2|z_3)$ of Z_1, Z_2 given $Z_3 = z_3$ is bivariate normal. Let $m_1^*, m_2^*; V_1^*, V_2^*; \Gamma^*$ denote the conditional means, variances and covariance respectively. Then

$$\begin{aligned} m_1^* &= (C_{12}C_{23} - C_{13}C_{22})z_3 / (C_{11}C_{22} - C_{12}^2), \\ m_2^* &= (C_{12}C_{13} - C_{11}C_{23})z_3 / (C_{11}C_{22} - C_{12}^2), \\ V_1^* &= C_{22}|C| / (C_{11}C_{22} - C_{12}^2), \quad V_2^* = C_{11}|C| / (C_{11}C_{22} - C_{12}^2), \\ \Gamma^* &= C_{12}|C| / (C_{11}C_{22} - C_{12}^2), \end{aligned} \quad (1.2)$$

where $|C|$ denotes the determinant of C .

2. Main result. Throughout the rest of this article, let $x(t), 0 \leq t \leq T$, denote a Gaussian process with mean function $m(t)$ and covariance function $R(s, t)$. Assume that $m(t)$ has a bounded first derivative for $0 \leq t \leq T$ and that $R(s, t)$ satisfies conditions

(R1) $R(s, t)$ is continuous in (s, t) and has continuous first partial derivatives in $\{(s, t): 0 \leq s \leq t \leq T\}$ (including one-sided partial derivatives at the boundary).

(R2) $|R(s + ds, t + dt) - R(s + ds, t) - R(s, t + dt) + R(s, t)| \leq \text{constant}|ds dt|$ for $s \neq t$.

(R3) $R(t, t)R(s, s) - R(s, t)^2 > 0$ for $s \neq t$ (this condition will be dropped later, see Remark 1).

Condition (R1) implies that $f(t)$, as defined by (0.2), exists and is bounded and continuous for $0 \leq t \leq T$. One can check easily that $f(t)$ is nonnegative.

THEOREM 2.1. *Let G be a polynomial or a bounded continuous function. Then the L^2 -limit of $\sum_k G(x(t_k))\Delta x(t_k)^2$, along any sequence of partitions $P_n = \{t_k\}$, $n \geq 1$, of $[0, T]$ whose mesh d_n goes to 0, exists and equals $\int_0^T G(x(t))f(t) dt$. (Recall $\Delta x(t_k) = x(t_{k+1}) - x(t_k)$.)*

We shall prove Theorem 2.1 for the case $m(t) \equiv 0$ only. For the case $m(t) \not\equiv 0$, the proof is similar to that of the special case $G(x) \equiv 1$ (see [1, pp. 524–525]). For each partition $P_n = \{t_k\}$, let $a_{ij} = E\{\Delta x(t_i)\Delta x(t_j)\}$. Then conditions (R1) and (R2) imply

$$\begin{cases} a_{ij} = O(\Delta t_i \Delta t_j), & i \neq j, \\ a_{kk} = f(t_k)\Delta t_k + o(\Delta t_k), \end{cases} \quad (2.1)$$

where the Landau notations $O(\cdot)$ and $o(\cdot)$ are uniform in i, j, k as $d_n \rightarrow 0$ and where $\Delta t_m = t_{m+1} - t_m$ for $m = i, j, k$.

Given $\varepsilon > 0$, condition (R3) implies that there exists $M_\varepsilon > 0$ such that, for $|t - s| \geq \varepsilon$,

$$R(t, t)R(s, s) - R(s, t)^2 \geq M_\varepsilon. \quad (2.2)$$

Let $I_\varepsilon = \{(i, j): |t_i - t_j| \geq \varepsilon, f(t_i) \geq \varepsilon, f(t_j) \geq \varepsilon\}$. For $(i, j) \in I_\varepsilon$, let $z_1 = \Delta x(t_i)$, $z_2 = \Delta x(t_j)$, $z_3 = x(t_i)$, $z_4 = x(t_j)$ and let $B = (b_{km})$ denote their covariance matrix as in §1. Then $b_{11} = a_{ii}$, $b_{22} = a_{jj}$, $b_{33}b_{44} - b_{34}^2 = R(t_i, t_i)R(t_j, t_j) - R(t_i, t_j)^2$. From the assumption on $R(s, t)$, it is clear that

$$b_{km} = \begin{cases} O(\Delta t_i), & k = 1, m = 3, 4; m = 1, k = 3, 4, \\ O(\Delta t_j), & k = 2, m = 3, 4; m = 2, k = 3, 4. \end{cases} \quad (2.3)$$

Therefore, after expanding each cofactor B_{km} , $1 \leq k, m \leq 4$, (2.1) and (2.3) imply that

$$\begin{aligned} B_{11} &= a_{jj}D + O(\Delta t_j^2), & B_{22} &= a_{ii}D + O(\Delta t_i^2), \\ B_{12} &= a_{ij}D + O(\Delta t_i \Delta t_j), & B_{km} &= O(\Delta t_i \Delta t_j), \quad k = 1, 2, m = 3, 4, \\ B_{11}B_{22} - B_{12}^2 &= (a_{ii}a_{jj} - a_{ij}^2)D^2 + O(\Delta t_i^2 \Delta t_j) + O(\Delta t_i \Delta t_j^2), \\ B_{12}B_{23} - B_{13}B_{22} &= O(\Delta t_i^2 \Delta t_j) = B_{12}B_{24} - B_{14}B_{22}, \\ B_{12}B_{13} - B_{11}B_{23} &= O(\Delta t_i \Delta t_j^2) = B_{12}B_{14} - B_{11}B_{24}, \\ |B| &= a_{ii}a_{jj}D + O(\Delta t_i^2 \Delta t_j^2) = (a_{ii}a_{jj} - a_{ij}^2)D + O(\Delta t_i^2 \Delta t_j^2), \end{aligned} \quad (2.4)$$

where $D = b_{33}b_{44} - b_{34}^2$. From (1.1), (2.1), (2.2), (2.4) and the fact that $f(t_i) \geq \varepsilon$, $f(t_j) \geq \varepsilon$, we see that

$$\begin{aligned} m_1 &= O(\Delta t_i)z_3 + O(\Delta t_i)z_4, & m_2 &= O(\Delta t_j)z_3 + O(\Delta t_j)z_4, \\ V_1 &= a_{ii} + O(\Delta t_i^2), & V_2 &= a_{jj} + O(\Delta t_j^2), & \Gamma &= O(\Delta t_i \Delta t_j). \end{aligned} \quad (2.5)$$

LEMMA 2.2. *Let $(i, j) \in I_\varepsilon$. Then*

$$\begin{aligned} E\{\Delta x(t_i)^2 \Delta x(t_j)^2 | x(t_i) = z_3, x(t_j) = z_4\} \\ = a_{ii}a_{jj} + O(\Delta t_i^2 \Delta t_j) + O(\Delta t_i \Delta t_j^2) + P_1(z_3, z_4) \end{aligned}$$

where $P_1(z_3, z_4)$ is a polynomial of z_3, z_4 of degree 4 with coefficients of order $\Delta t_i^2 \Delta t_j$ or $\Delta t_i \Delta t_j^2$.

PROOF. From Property 1.2 and Property 1.3, we obtain

$$\begin{aligned} E \{ \Delta x(t_i)^2 \Delta x(t_j)^2 | x(t_i) = z_3, x(t_j) = z_4 \} \\ = \int \int_{R^2} z_1^2 z_2^2 N(z_1, z_2 | z_3, z_4) dz_1 dz_2 \\ = V_1 V_2 + 2\Gamma^2 + m_1^2 V_2 + m_2^2 V_1 + 4m_1 m_2 \Gamma + m_1^2 m_2^2. \end{aligned} \quad (2.6)$$

Lemma 2.2 follows clearly from (2.6), (2.5) and (2.1).

By using a similar argument and the identity $EX^2 = \text{Var}(X) + (EX)^2$, we can show

LEMMA 2.3. Let $(i, j) \in I_e$. Then

$$\begin{aligned} E \{ \Delta x(t_i)^2 | x(t_i) = z_3, x(t_j) = z_4 \} &= a_{ii} + O(\Delta t_i^2) + P_2(z_3, z_4), \\ E \{ \Delta x(t_j)^2 | x(t_i) = z_3, x(t_j) = z_4 \} &= a_{jj} + O(\Delta t_j^2) + P_3(z_3, z_4), \end{aligned}$$

where P_2, P_3 are homogeneous polynomials of degree 2. All of their coefficients are of order Δt_i^2 and Δt_j^2 respectively.

Let $e_{ij} = G(x(t_i))G(x(t_j))(\Delta x(t_i)^2 - a_{ii})(\Delta x(t_j)^2 - a_{jj})$. Then

COROLLARY 2.4. Let G be as in Theorem 2.1. Then

$$\lim_{n \rightarrow \infty} \sum_{(i,j) \in I_e} E \{ e_{ij} \} = 0.$$

PROOF. Lemma 2.2 together with Lemma 2.3 imply that, for $(i, j) \in I_e$,

$$\begin{aligned} E \{ e_{ij} \} &= E \{ E \{ e_{ij} | x(t_i), x(t_j) \} \} \\ &= E \{ G(x(t_i))G(x(t_j)) [O(\Delta t_i^2 \Delta t_j) + O(\Delta t_i \Delta t_j^2) + P_4(x(t_i), x(t_j))] \}, \end{aligned} \quad (2.7)$$

where $P_4(z_3, z_4)$ is a polynomial of z_3, z_4 of degree 4 with all its coefficients of order $\Delta t_i^2 \Delta t_j$ or $\Delta t_i \Delta t_j^2$. The assumption on G implies that $E \{ G(x(s))G(x(t)) [x(s)]^p [x(t)]^q \}$ is bounded for fixed integers $p, q > 0$. Therefore, $E \{ e_{ij} \} = O(\Delta t_i^2 \Delta t_j) + O(\Delta t_i \Delta t_j^2)$. Summing up (2.7) over $(i, j) \in I_e$ and letting $n \rightarrow \infty$, Corollary 2.4 follows.

LEMMA 2.5. Let $I_1 = \{(i, j): |t_i - t_j| < \varepsilon\}$. Then

$$\overline{\lim}_{n \rightarrow \infty} \left| \sum_{(i,j) \in I_1} E \{ e_{ij} \} \right| \leq \text{constant} \cdot \varepsilon.$$

PROOF. By applying the Cauchy-Schwarz inequality twice, we obtain

$$\begin{aligned} |E \{ e_{ij} \}| &\leq \left\{ E \left[G(x(t_i))^2 G(x(t_j))^2 \right] \right\}^{1/2} \left\{ E \left[\Delta x(t_i)^2 - a_{ii} \right]^4 \right\}^{1/4} \\ &\quad \cdot \left\{ E \left[\Delta x(t_j)^2 - a_{jj} \right]^4 \right\}^{1/4}. \end{aligned} \quad (2.8)$$

Let $M = \sup\{[E\{G(x(t))^2G(x(s))^2\}]^{1/2}; 0 \leq s, t \leq T\}$. Then M is finite. Property 1.1 and (2.8) imply that

$$|E\{e_{ij}\}| \leq Mc_4^2 a_{ii} a_{jj} = O(\Delta t_i \Delta t_j). \quad (2.9)$$

Therefore,

$$\left| \sum_{(i,j) \in I_1} E\{e_{ij}\} \right| \leq \sum_i \sum_{\{j: |t_j - t_i| < \varepsilon\}} |E\{e_{ij}\}| \leq O(1) \cdot \varepsilon.$$

Lemma 2.5 follows by letting $n \rightarrow \infty$ in the above inequality.

LEMMA 2.6. Let $I_2 = \{(i, j): f(t_i) < \varepsilon \text{ or } f(t_j) < \varepsilon\}$. Then

$$\overline{\lim}_{n \rightarrow \infty} \left| \sum_{(i,j) \in I_2} E\{e_{ij}\} \right| \leq \text{constant} \cdot \varepsilon.$$

PROOF. Let $(i, j) \in I_2$ and suppose that $f(t_i) < \varepsilon$. From (2.1) and the first inequality of (2.9), we see that

$$|E\{e_{ij}\}| \leq Mc_4^2 a_{ii} a_{jj} \leq \varepsilon O(\Delta t_i \Delta t_j) + o(\Delta t_i \Delta t_j). \quad (2.10)$$

Note that (2.10) also holds if $(i, j) \in I_2$ and $f(t_j) < \varepsilon$. Hence, after summing up (2.10) over $(i, j) \in I_2$ and passing $n \rightarrow \infty$, Lemma 2.6 follows.

From Corollary 2.4, Lemma 2.5 and Lemma 2.6, we see that

$$L^2\text{-}\lim_{n \rightarrow \infty} \sum_k G(x(t_k)) \Delta x(t_k)^2 = L^2\text{-}\lim_{n \rightarrow \infty} \sum_k G(x(t_k)) a_{kk} \quad (2.11)$$

if the L^2 -limit in the R.H.S. of (2.11) exists. This limit exists since $E\{G(x(s))G(x(t))\}$ is continuous in (s, t) , $a_{kk} = f(t_k)\Delta t_k + o(\Delta t_k)$, and $f(t)$ is bounded continuous. This limit is denoted by $\int_0^T G(x(t))f(t) dt$. This completes the proof of Theorem 2.1.

REMARK 1. Condition (R3) is equivalent to the statement that $x(s)$, $x(t)$ are linearly independent for $s \neq t$. In the proofs of Lemmas 2.2–2.3, if $x(t_i)$, $x(t_j)$ are linearly dependent, then we use Property 1.4 instead of Property 1.3. It would not affect the result of Lemmas 2.2–2.3. Therefore, Corollary 2.4 and, hence, Theorem 2.1 are still valid if we drop condition (R3).

REMARK 2. From the process of the proof of Theorem 2.1, we see that condition (R2) can be weakened to the statement that "For any $\varepsilon > 0$, there exists M_ε such that

$$|R(t + dt, s + ds) - R(t + dt, s) - R(t, s + ds) + R(t, s)| \leq M_\varepsilon |dtds|$$

uniformly for $s \neq t$ except on a set with Lebesgue measure smaller than ε ". A sufficient condition for this is that $R(s, t)$ is of bounded variation.

3. Application. Let $H(y)$ be a function of y . We shall denote $\int_s^t H(x(r)) dx(r)$ as the L^2 -limit of $\sum_k H(x(t_k)) \Delta x(t_k)$ along any sequence of partitions of $[s, t]$ whose mesh goes to 0, when the limit exists.

Let $U(x)$ be a function having bounded continuous $U'(x)$, $U''(x)$ and bounded $U'''(x)$. Then for $0 \leq s < t \leq T$,

$$U(x(t)) - U(x(s)) = \sum_{k=0}^{n-1} U'(x(t_k))\Delta x(t_k) + \frac{1}{2} \sum_{k=0}^{n-1} U''(x(t_k))\Delta x(t_k)^2 + \frac{1}{6} \sum_{k=0}^{n-1} U'''(y_k)\Delta x(t_k)^3 = J_1 + J_2 + J_3, \quad (3.1)$$

where $t_k = s + k(t-s)/n$, $0 \leq k \leq n$, $\Delta x(t_k) = x(t_{k+1}) - x(t_k)$, $0 \leq k \leq n-1$, and where y_k is between $x(t_k)$ and $x(t_{k+1})$ for each k . Using the fact that $E\Delta x(t_k)^{2m} = O(a_{kk}^m) = O(\Delta t_k^m)$ uniformly in k for each $m \geq 1$ and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} E\{J_3^2\} &\leq \|U\|_\infty^2 \lim_{n \rightarrow \infty} \sum_{k,j=0}^{n-1} E\{|\Delta x(t_k)|^3 |\Delta x(t_j)|^3\} \\ &\leq \|U\|_\infty^2 \lim_{n \rightarrow \infty} \sum_{k,j=0}^{n-1} O(\Delta t_k^{3/2}) O(\Delta t_j^{3/2}) = 0, \end{aligned} \quad (3.2)$$

where $\|U\|_\infty$ is the supremum norm of U . Letting $n \rightarrow \infty$ in (3.1), Theorem 2.1 and (3.2) imply that J_1 converges in L^2 . Denoting the limit by $\int_s^t U'(x(r)) dx(r)$, we have

$$U(x(t)) - U(x(s)) = \int_s^t U'(x(r)) dx(r) + \frac{1}{2} \int_s^t U''(x(r)) f(r) dr. \quad (3.3)$$

A slight modification of the above proof shows that (3.3) holds also for the polynomial $U(x)$. The formal differential rule is

THEOREM 3.1. *Let $U(t, x)$, $0 \leq t \leq T$, $x \in R$, have bounded continuous $U'_1(t, x)$, $U'_2(t, x)$, $U''_{22}(t, x)$ and bounded $U'''_{222}(t, x)$ or be a polynomial of x with all the coefficients continuously differentiable with respect to t . Then, for $0 \leq s < t \leq T$*

$$\begin{aligned} U(t, x(t)) &= U(s, x(s)) + \int_s^t U'_2(r, x(r)) dx(r) \\ &\quad + \int_s^t \left\{ U'_1(r, x(r)) + \frac{1}{2} U''_{22}(r, x(r)) f(r) \right\} dr. \end{aligned} \quad (3.4)$$

REMARK 3. The differential formula for a Gaussian process $x(t)$ given in [3, (8.1)] looks different from (3.4). This is because the definitions of $\int_s^t U(x(r)) dx(r)$ are different. Note that in [3, (8.1)], $U(t, x)$ is ∞ -differentiable w.r.t. x and in Theorem 3.1, it is thrice differentiable w.r.t. x only.

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