TRANSITIVITY AND THE γ -SPACE CONJECTURE IN ORDERED SPACES

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ABSTRACT. Each generalized ordered γ -space is nonarchimedean quasimetrizable. Moreover, each generalized ordered space is 3-transitive, i.e. for each neighbourhood U there is a transitive neighbourhood $V \subset U^3$.

I. Introduction.

1. The concept of a *quasimetric* is quite old [N1], [W]. It is a generalized metric d with the triangle axiom $d(x, z) \leq d(x, y) + d(y, z)$, but generally without the symmetry axiom $d(x, y) = d(y, x)$. If the triangle axiom is strengthened to " $d(x, y) \leq \max\{d(x, y), d(y, z)\}$ ", then d is a *nonarchimedean quasimetric* [FL1], [K]. If it is relaxed to " $d(x, y_n) \rightarrow 0$ and $d(y_n, z_n) \rightarrow 0$ implies $d(x, z_n) \rightarrow 0$ ", then d is a γ -metric and a space with a γ -metric is a γ -space [N2], [H].

Obviously, a nonarchimedean quasimetric space is quasimetric and a quasimetric space is a γ -space. However, a quasimetric space need not be nonarchimedean quasimetrizable [K], while the problem whether each γ -space is quasi-metrizable, listed as Classic Problem VIII in [T], is open and only partial solutions have been obtained [G], [J1], [B], [F]. In [B] it was shown that if a linearly ordered space X has a σ -discrete dense set then every generalized ordered space constructed on X is quasimetrizable if it is a γ -space.

It will be proved here that each generalized ordered γ -space is nonarchimedean quasimetrizable.

2. That result will follow from more general results concerned with the transitivity of ordered spaces.

Recall that a binary relation on a set X is any subset R of $X \times X$. Given a binary relation R on X we define $R\{x\} = \{y \in X \mid (x, y) \in R\}$. Conversely, we can construct a binary relation on X by specifying each of the sets $R\{x\}$ and then letting $R = \bigcup \{\{x\} \times R\{x\} \mid x \in X\}$. Throughout the rest of this paper, we will systematically confuse binary relations as subsets of X^2 and binary relations which are obtained by specifying subsets $R\{x\}$ of X , as before. For a binary relation R on X and any subset $G \subset X$, we define $RG = R(G) = \bigcup \{R\{x\} \mid x \in G\}$. Given two binary relations R and S on X , a new binary relation $S \circ R$ is defined by $S \circ R = \{(x, z) \mid \text{for some } y \in X, (x, y) \in R \text{ and } (y, z) \in S\}$. Equivalently, $S \circ R$ can be obtained by defining $(S \circ R)\{x\} = S(R\{x\})$. If R is a binary relation, then

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R^2 denotes $R \circ R$, $R^3 = R \circ R^2$, etc. A binary relation is *transitive* if $R^2 \subset R$, i.e., if $z \in R\{x\}$ whenever $z \in R\{y\}$ and $y \in R\{x\}$.

A binary relation U on a space X is called *neighbournet in X* provided that each $U\{x\}$ is a neighbourhood of x . If U is a neighbourhood of the diagonal in $X \times X$, then it is a neighbournet in X , but the converse need not be right.

A sequence $\langle U_n \rangle$ of neighbournets is called *basic* if for each $x \in X$, $\{U_n\{x\} \mid n \geq 1\}$ is a neighbourhood base at x , and the sequence is called a *normal sequence* if $U_{n+1}^2 \subset U_n$ for each n . A neighbournet U is said to be *normal* if it is a member of a normal sequence of neighbournets [J2]. Clearly, if d is a quasimetric for X and if $U_n = \{(x, y) \in X^2 \mid d(x, y) < 1/2^n\}$, then $\langle U_n \rangle$ is a basic, normal sequence. Indeed, the following characterization is known [FL1], [K], [LF], [J2].

PROPOSITION 1. (a) *A space X is quasimetrizable if and only if X has a basic, normal sequence of neighbournets.*

(b) *A space X is nonarchimedean quasimetrizable if and only if X has a basic sequence of transitive neighbournets.*

(c) *A space X is a γ -space if and only if there is a sequence $\langle U_n \rangle$ of neighbournets in X such that $\langle U_n^2 \rangle$ is a basic sequence. $\square$*

Concerning (b) of Proposition 1, we remark that any decreasing sequence of transitive neighbournets is a normal sequence; in part (c), if $\langle U_n^2 \rangle$ is basic, then so is $\langle U_n^k \rangle$ for each fixed $k \geq 1$.

A γ -space X is quasimetrizable (resp., nonarchimedean quasimetrizable) if for some $k \geq 1$ and for each neighbournet U , there is a normal (resp. transitive) neighbournet V with $V \subset U^k$.

A space is called *transitive* if for each normal neighbournet U there is a transitive neighbournet $V \subset U$, and is *k -transitive* if for each neighbournet U there is a transitive neighbournet $V \subset U^k$ [FL2], [FL3]. It follows that k -transitive γ -spaces ($k > 1$) as well as transitive quasimetric spaces are nonarchimedean quasimetrizable.

Obviously, a k -transitive space is transitive as well as $(k + 1)$ -transitive. A space is seldom k -transitive (cf. later Remark). Quasimetric spaces which are not nonarchimedean quasimetrizable fail to be transitive. Metric spaces are transitive [J2]. These and other results on transitive spaces can be found in [J2], [FL3]. It is not known, however, if other common types of spaces, for example, compact spaces or nonarchimedean quasimetrizable spaces, are transitive.

The main result of this paper states that all generalized ordered spaces are 3-transitive.

3. A Hausdorff space is called *generalized ordered space* (= GO-space) if the underlying set is linearly ordered and some collection of intervals forms a base for the topology; i.e. for each $x \in X$ either all $]a, b[$, with $a < x < b$; or all $]a, x]$, with $a < x$; or all $[x, b[$, with $x < b$; or $\{x\}$ form a base of neighbourhoods [Lu].

Let X be a space, and U be a neighbournet in X . Define a new neighbournet U^+ by setting $U^+\{x\} = \bigcap \{UG \mid G \text{ is a neighbourhood of } x\}$. One has $U \subset U^+ \subset UV$ for each neighbournet V in X , and hence $U^k \subset (U^k)^+ \subset U^{k+1}$ for each $k \geq 1$. If G is an open set then $U^+(G) = U(G)$.

II. 2-transitivity in Sorgenfrey spaces. A GO-space is called *Sorgenfrey* if each $[x, \rightarrow[$ is open in X . Let X be a Sorgenfrey space and let U be a neighbourhood in X . Given a set $Y \subset X$ and a point $r \in X$ we put $R(Y, r, U) = \{x < r \mid [x, r[\subset Y \cap U^+\{x\}\}$. A family $\langle\langle R_\alpha, r_\alpha \rangle\rangle_{\alpha \in A}$, where A is an initial set of ordinals, is called *U-regular in X* provided that $R_\alpha = R(X - \bigcup_{\beta < \alpha} R_\beta, r_\alpha, U)$ for each α .

LEMMA 1. *Let X be a Sorgenfrey space, U be a neighbourhood and $\langle\langle R_\alpha, r_\alpha \rangle\rangle_{\alpha < \alpha_0}$ be a U -regular family. Then*

- (a) *the sets R_α are disjoint,*
- (b) *if $x \in R_\alpha$ then $[x, r_\alpha[\subset U^+\{x\}$,*
- (c) *if $x \in R_\alpha, y \in R_\alpha$ and $y \in [x, r_\alpha[$ then $\alpha' > \alpha$ and $r_{\alpha'} < r_\alpha$,*
- (d) *for each $\alpha \leq \alpha_0$ the set $\bigcup_{\beta < \alpha} R_\beta$ is closed in X .*

PROOF. (a) and (b) are obvious.

(c) Since $y \in [x, r_\alpha[\subset X - \bigcup_{\beta < \alpha} R_\beta$ we get $\alpha' > \alpha$. If $r_{\alpha'} > r_\alpha$ we have $[y, r_\alpha[\subset [y, r_{\alpha'}[\subset U^+\{y\}$ and $[y, r_\alpha[\subset [x, r_\alpha[\subset X - \bigcup_{\beta < \alpha} R_\beta$; hence $y \in R_\alpha$, i.e. $\alpha' = \alpha$.

(d) Suppose that α is the smallest ordinal $< \alpha_0$ such that $\bigcup_{\beta < \alpha} R_\beta$ is not closed and let $x \in \text{cl}(\bigcup_{\beta < \alpha} R_\beta) - \bigcup_{\beta < \alpha} R_\beta$.

If $\alpha = \beta + 1$, then $\text{cl} \bigcup_{\beta < \alpha} R_\beta = \text{cl}(\bigcup_{\gamma < \beta} R_\gamma) \cup \text{cl} R_\beta = \bigcup_{\gamma < \beta} R_\gamma \cup \text{cl} R_\beta$. The set $\bigcup_{\gamma < \beta} R_\gamma$ is closed and hence $x \in \text{cl} R_\beta$. Let us show that $x \in R_\beta$. Indeed, if G is an open neighbourhood of x and $G \subset [x, \rightarrow[$ then one has $[x, r_\beta[\subset \bigcup \{[y, r_\beta[\mid y \in G \cap R_\beta\} \subset \bigcup \{[y, r_\beta[\mid y \in R_\beta\} \subset X - \bigcup_{\gamma < \beta} R_\gamma$ by definition of R_β . On the other hand, by the same definition,

$$\bigcup \{[y, r_\beta[\mid y \in G \cap R_\beta\} \subset U^+(G \cap R_\beta) \subset U^+(G) = U(G)$$

since G is an open set. Hence

$$\begin{aligned} [x, r_\beta[&\subset \left(X - \bigcup_{\gamma < \beta} R_\gamma\right) \cap (\cap \{U(G) \mid G \text{ is a neighbourhood of } x\}) \\ &\subset \left(X - \bigcup_{\gamma < \beta} R_\gamma\right) \cap U^+\{x\}, \end{aligned}$$

i.e. $x \in R(X - \bigcup_{\gamma < \beta} R_\gamma, r_\beta, U) = R_{x_\beta}$. This contradicts $x \notin \bigcup_{\gamma < \alpha} R_\gamma$.

If $\alpha = \sup\{\beta \mid \beta < \alpha\}$, we pick $y \in R_{\alpha'}$ with $\alpha' < \alpha$, $[x, y[\subset U^+\{x\}$, and $z \in R_\beta$ with the smallest β , $\alpha' < \beta < \alpha$ such that $[x, z[\subset X - \bigcup_{\gamma < \alpha} R_\gamma$. Obviously $[x, z[\subset X - \bigcup_{\gamma < \beta} R_\gamma$. Since $y \in R_{\alpha'} \subset \bigcup_{\gamma < \beta} R_\gamma$ we have $y \notin [x, z[$. We also have $y \in [z, r_\beta[$ by (c) since $z \in R_\beta, y \in R_{\alpha'}$ and $\alpha' < \beta$. Hence $[x, r_\beta[\subset [x, y[\subset U^+\{x\}$ and $[x, r_\beta[= [x, z[\cup [z, r_\beta[\subset X - \bigcup_{\gamma < \beta} R_\gamma$. Hence

$$x \in R\left(X - \bigcup_{\gamma < \beta} R_\gamma, r_\beta, U\right) = R_\beta$$

and this contradicts $x \notin \bigcup_{\gamma < \alpha} R_\gamma$. $\square$

Set $R_U = \{x \mid \exists r > x, [x, r[\subset U^+\{x\}\}$. Each $R(Y, r, U) \subset R_U$ hence if $\langle R_\alpha, r_\alpha \rangle$ is a U -regular family then $\bigcup_\alpha R_\alpha \subset R_U$. A U -regular family $\langle\langle R_\alpha, r_\alpha \rangle\rangle$ is called *maximal* if $\bigcup_\alpha R_\alpha = R_U$.

LEMMA 2. *For each Sorgenfrey space X and neighbourhood U in X there exists a maximal U -regular family in X .*

PROOF. Suppose that we already have a regular family $\langle\langle R_\alpha, r_\alpha \rangle\rangle$ and $\bigcup_{\beta < \alpha} R_\beta \neq R_U$. Extend the family as follows. Let $x \in R_U - \bigcup_{\beta < \alpha} R_\beta$. The set $\bigcup_{\beta < \alpha} R_\beta$ is closed by Lemma 1(d). Hence for some $r_\alpha \in X$ we have

$$[x, r_\alpha[\subset U\{x\} \cap \left(X - \bigcup_{\beta < \alpha} R_\beta \right).$$

Then $R_\alpha = R(X - \bigcup_{\beta < \alpha} R_\beta, r_\alpha, U) \neq \emptyset$ since $x \in R_\alpha$. $\square$

THEOREM 1. *In a Sorgenfrey space, for each neighbournet U there exists a transitive neighbournet $V \subset U^+$.*

PROOF. Let $\langle\langle R_\alpha, r_\alpha \rangle\rangle$ be a maximal U -regular family in X by Lemma 2. For $x \in X - R_U$ set $V\{x\} = \{x\}$. By Lemma 1(a), $\langle R_\alpha \rangle$ is a partition of R_U . For $x \in R_\alpha$ set $V\{x\} = [x, r_\alpha[$. Then $V\{x\} \subset U^+\{x\}$ by Lemma 1(b). If $x \in R_\alpha$ and $y \in R_\alpha$, $y \in V\{x\} \subset [x, r_\alpha[$; then by Lemma 1(c) $r_\alpha < r_\alpha$ and $V\{y\} = [y, r'_\alpha[\subset [x, r_\alpha[= V\{x\}$; hence V is transitive. $\square$

Since $U^+ \subset U^2$ we get

THEOREM 2. *Each Sorgenfrey space is 2-transitive.* $\square$

III. 3-transitivity in GO-spaces. Let X be a GO-space. Associate with X two spaces X_r and X_l on the underlying set of X . The topology of X_r (X_l) is a supremum of the topology of X and that generated by all intervals $[x, y[$ ($]x, y]$). X_r is a Sorgenfrey space with respect to the order of X and X_l is a Sorgenfrey space with respect to the opposite order.

Let U be a neighbournet in X . Set $U_r\{x\} = U\{x\} \cap [x, \rightarrow[$ and $U_l\{x\} = U\{x\} \cap]\leftarrow, x]$. Then U_r (U_l) is a neighbournet in X_r (X_l).

PROPOSITION 2. *Let X be a GO-space and let U be a neighbournet in X such that for some neighbournet W , $U = W^2$ where W_r and W_l are transitive neighbournets in X_r and X_l . Then there exists a transitive neighbournet $V \subset U^+$.*

PROOF. If we define a transitive neighbournet V^R in X such that $V_r^R \subset U^+$ and, similarly dealing with the opposite order, a transitive neighbournet V^L in X such that $V_l^L \subset U^+$, we shall set $V = V^L \cap V^R$. Then V will be a transitive neighbournet in X and $V = V_l \cup V_r \subset V^L \cup V_r^R \subset U^+$, and the proof will be completed.

So let us define V^R .

Note that U , which is a neighbournet in X , can be considered as a neighbournet in X_r . Apply the operation $^+$ to U in X_r . Denote the result by $U_{X_r}^+$. We have $U_{X_r}^+ \subset U^+$. The space X_r is a Sorgenfrey space and hence by Lemma 2 there exists a maximal U -regular family $\langle\langle R_\alpha, r_\alpha \rangle\rangle$ in X_r .

Set $\tilde{V}\{x\} =]\leftarrow, x]$ when $x \notin R_U$ and $\tilde{V}\{x\} =]\leftarrow, r_\alpha[$ when $x \in R_\alpha$, and $P(x) = \{y \in \tilde{V}\{x\} \mid \tilde{V}\{y\} \not\subset \tilde{V}\{x\}\}$.

Now define $V^R\{x\} = \tilde{V}\{x\} - P(x)$. Since $y \in V^R\{x\}$ if and only if $\tilde{V}\{y\} \subset \tilde{V}\{x\}$, V^R is transitive.

$V_r^R\{x\}$ is either $\{x\}$ or for $x \in R_\alpha$, $V_r^R\{x\} \subset [x, r_\alpha[\subset U_{X_r}^+\{x\} \subset U^+\{x\}$.

It remains to show that V^R is a neighbourhood in X . We will use the fact that $U = W^2$, where W is a neighbourhood in X such that W_l and W_r are transitive. We shall need the following notes.

$$(W^2)_r = (W \circ W_l)_r. \quad (i)$$

Indeed $(W^2)_r = (W \circ (W_l \cup W_r))_r = (W \circ W_l)_r \cup (W \circ W_r)_r = (W \circ W_l)_r \cup ((W_l \cup W_r) \circ W_r)_r = (W \circ W_l)_r \cup (W_l \circ W_r)_r \cup (W_r^2) = (W \circ W_l)_r \cup W_r \cup W_r$, since $(W_l \circ W_r)_r = W_r$ and W_r is transitive.

$$\text{If } G \subset X, x \in X \text{ and for each } y \in G, y \leq x \text{ then } W^2(G) \cap [x, \rightarrow[= ((W \circ W_l)(G)) \cap [x, \rightarrow[. \quad (ii)$$

Indeed $W^2(G) \cap [x, \rightarrow[= (\cup_{y \in G} W^2\{y\}) \cap [x, \rightarrow[= (\cup_{y \in G} W^2\{y\} \cap [y, \rightarrow]) \cap [x, \rightarrow[= (\cup_{y \in G} (W^2)_r\{y\}) \cap [x, \rightarrow[= (W^2)_r(G) \cap [x, \rightarrow[$. By (i) the last set is $(W \circ W_l)_r(G) \cap [x, \rightarrow[= (\cup_{y \in G} (W \circ W_l)\{y\} \cap [y, \rightarrow]) \cap [x, \rightarrow[= (\cup_{y \in G} (W \circ W_l)\{y\}) \cap [x, \rightarrow[= ((W \circ W_l)(G)) \cap [x, \rightarrow[$.

Let us show now that V^R is a neighbourhood, i.e. $V^R\{x\} = \tilde{V}\{x\} - P(x)$ is a neighbourhood of x , i.e. $x \notin \text{cl } P(x)$. Note that $P(x) \subset]\leftarrow, x]$. Otherwise there is $y \in P(x)$, $x \leq y$. If $x \notin R_{U_l}$, $y = x$; hence $\tilde{V}\{y\} = \tilde{V}\{x\}$. If $x \in R_{\alpha}$ then $x < y < r_{\alpha}$ and by Lemma 1(c) $\tilde{V}\{y\} \subset \tilde{V}\{x\}$.

Suppose that $x \in \text{cl } P(x)$. Since $P(x) \subset]\leftarrow, x]$, there exists $y \in P(x)$ with $[y, x] \subset W_l\{x\}$, $y \in R_{\alpha}$ and $x < r_{\alpha}$. Let us show that this implies $x \in R_{\alpha}$. We have $[x, r_{\alpha}] \subset [y, r_{\alpha}] \subset X_r - \cup_{\beta < \alpha} R_{\beta}$. We also have $[x, r_{\alpha}] = [y, r_{\alpha}] \cap [x, \rightarrow[\subset U_{X_r}^+\{y\} \cap [x, \rightarrow[$. The set $G = V_l\{x\} \supset [y, x]$ is a neighbourhood of y in X_r . Hence $U_{X_r}^+\{y\} \cap [x, \rightarrow[\subset U(G) \cap [x, \rightarrow[= (W^2)(G) \cap [x, \rightarrow[$. By (ii) we have $W^2(G) \cap [x, \rightarrow[= ((W \circ W_l)(G)) \cap [x, \rightarrow[= ((W \circ W_l)(W_l\{x\})) \cap [x, \rightarrow[$ since $G = W_l\{x\}$. Since W_l is transitive, $((W \circ W_l)\{x\}) \cap [x, \rightarrow[= ((W \circ W_l)\{x\}) \cap [x, \rightarrow[= (W \circ W_l)_r\{x\}$. By (i) $(W \circ W_l)_r\{x\} = (W^2)_r\{x\} = (W^2)_r\{x\} \subset U\{x\}$. Hence $[x, r_{\alpha}] \subset (X_r - \cup_{\beta < \alpha} R_{\beta}) \cap U\{x\}$, i.e., $x \in R(X_r - \cup_{\beta < \alpha} R_{\beta}, r_{\alpha}, U) = R_{\alpha}$, and $\tilde{V}\{x\} =]\leftarrow, r_{\alpha}] = \tilde{V}\{y\}$, hence $y \notin P(x)$. This contradiction proves that $x \notin \text{cl } P(x)$ and hence V^R is a neighbourhood in X . $\square$

THEOREM 3. *In a GO-space, for each neighbourhood U there exists a transitive neighbourhood $V \subset U^{2+}$.*

PROOF. Suppose that each $U\{x\}$ is open in X . Apply Theorem 1 to the Sorgenfrey space X_r and the neighbourhood U in X_r . We get a transitive neighbourhood W^R in X_r , such that $W^R \subset U_{X_r}^+ \subset U^+$ ($U_{X_r}^+$ is the result of operation $^+$ in X_r). Similarly we find a transitive neighbourhood W^L in X_l such that $W^L \subset U^+$. We may assume that $W^R = W_r^R$, and $W^L = W_l^L$ and set $W = W^L \cup W^R$. Then W is a neighbourhood in X , and $W_r = W^R$ and $W_l = W^L$ are transitive. Now apply Proposition 2 to $\tilde{U} = W^2$. We get a transitive neighbourhood $V \subset \tilde{U}^+ = W^{2+} \subset (U^+)^{2+}$. To show that $(U^+)^{2+} = U^{2+}$, consider an open G in X . Then $U(G)$ is open too and $(U^+)^2(G) = U^+(U^+(G)) = U^+(U(G)) = U(U(G)) = U^2(G)$. $\square$

Since $U^{2+} \subset U^3$ we get

THEOREM 4. *Each GO-space is 3 transitive.* $\square$

It was explained in the Introduction that k -transitive γ -spaces are nonarchimedean quasimetrizable spaces; therefore we have

THEOREM 5. *Each GO γ -space is nonarchimedean quasimetrizable.* $\square$

IV. Remarks and examples. (i) Sorgenfrey spaces need not be 1-transitive and (ii) GO-spaces need not be 2-transitive. In this sense Theorems 1 and 3 cannot be improved.

To show (i) take the Sorgenfrey line S which has the reals as the underlying set and whose topology has the intervals $[a, b[$ as a base. The neighbourhood U defined by $U\{x\} = [x, x + 1[$ for each irrational x and $U\{x_n\} = [x_n, x_n + 1/n[$ for the n th rational x_n does not contain any transitive neighbourhood V . Otherwise, set $S_i = \{x \in S \mid [x, x + 1/i[\subset V\{x\}\}$. Some S_i is dense in an open interval I . Pick a rational $x_n \in I$ with $U\{x_n\} \subset I$ and $n > i$ and let $x \in X_i \cap V\{x_n\}$; then $[x, x + 1/i[\subset V\{x\} \subset V\{x_n\} \subset U\{x_n\} \subset [x_n, x_n + 1/n[$, and hence $n < i$.

To show (ii) take the Engelking-Lutzer line X which has the reals as the underlying set, and which has the intervals $[a, b[$ with rational a and $[c, d[$ with irrational d as a base for the topology [ELu]. Consider a neighbourhood U such that $U\{x\} =]x - 1, x[$ for each irrational x and $U\{x_n\} \subset [x_n, x_n + 1/n[$ for the n th rational x_n and where $U\{x_n\}$ is either disjoint from or contained in each of $U\{x_1\}, \dots, U\{x_{n-1}\}$. Then $U^2\{x_n\} \subset]x_n - 1, x_n + 1/n[$, and U^2 does not contain any transitive neighbourhood V . Otherwise set $X_i = \{x \in X \mid [x, x + 1/i[\subset V\{x\}\}$, $X_0 = X - \bigcup_{i \geq 1} X_i$, and $\tilde{X}_i = \{x \in X_0 \mid]x - 1/i, x[\subset V\{x\}\}$. One of the sets $X_i, \tilde{X}_i$ is dense in an open interval I . Let us show first that it cannot be $\tilde{X}_i$. If $\tilde{X}_i$ would be dense in I , pick a rational $x \in I$ and let $[x, x + 2/k[\subset V\{x\}$, $k > i$. Now pick $y \in \tilde{X}_i \cap [x, x + 1/k[$. Then $y - 1/i < x < y < x + 1/k$, and $]y - 1/i, y[\subset V\{y\}$ and since V is transitive, $V\{x\} \subset V\{y\}$. Hence $]y, y + 1/k[\subset [x, x + 2/k[\subset V\{y\}$, i.e., $y \in X_k$ and that contradicts $y \in \tilde{X}_i$. Therefore, some X_i must be dense in I . Pick a rational $x_n \in I$ with $n > i$ and let $x \in X_i \cap [x_n, x_n + 1/n[\cap V\{x_n\}$. Since V is transitive, $[x, x + 1/i[\subset V\{x\} \cap [x_n, \rightarrow[\subset V\{x_n\} \cap [x_n, \rightarrow[\subset U^2\{x_n\} \cap [x_n, \rightarrow[\subset]x_n, \rightarrow[\subset]x_n, \rightarrow[\cap [x_n, \rightarrow[= [x_n, x_n + 1/n[$ and hence $n < i$. Hence no set X_i can be dense in I and this contradiction completes the proof.

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