ABSOLUTE C^* -EMBEDDING OF EXTREMALLY DISCONNECTED SPACES

ALAN DOW

ABSTRACT. An extremally disconnected space X is C^* -embedded in each extremally disconnected space in which it is embedded iff X is weakly Lindelöf or almost compact.

1. Introduction. All spaces under discussion are Tychonoff. A space X is *extremally disconnected*, denoted ED, if disjoint open subsets of X have disjoint closures in X . A space is *weakly Lindelöf* if every open cover has a countable subcollection whose union is dense [CHN]. It is well known and easy to prove that a weakly Lindelöf subspace X of an ED space Y is C^* -embedded in Y . It is also well known that almost compact subspaces are always C^* -embedded. In this paper we prove the converse; if an ED space X is not weakly Lindelöf nor almost compact then there is an ED space Y such that X can be embedded in Y so that it is not C^* -embedded. This question was raised by C. E. Aull [A]. Our method of proof is a generalization of a technique of Balcar and Simon [BS].

2. Preliminaries. We shall need some facts regarding ED spaces and the absolute of a space.

THEOREM 2.1 (1H AND 6M OF [GJ]). *The following are equivalent for a space X .*

- (1) X is ED.
- (2) βX is ED.
- (3) Each dense subspace of X is ED.
- (4) Each open subset of X has open closure in X . $\square$

Let Y be any regular space. Let $R(Y)$ be the boolean algebra of regular closed subsets of Y and let $S(R(Y))$ be the Stone space of $R(Y)$. The *absolute* of Y , denoted $E(Y)$, is the subspace of $S(R(Y))$ consisting of all convergent maximal filters of $R(Y)$. The map $k: E(Y) \rightarrow Y$ which is the canonical map from $E(Y)$ to Y takes each convergent maximal filter of $R(Y)$ to its limit in Y . This map is continuous, perfect and irreducible. The space $E(Y)$ is extremally disconnected. The reader may refer to [Wo] for more details concerning the absolute.

A base for the clopen subsets of $E(Y)$ is $\{cl_{E(Y)} k^+[\text{int } U]: U \in R(Y)\}$. It can be shown that $cl_{E(Y)} k^+[\text{int } U]$, for $U \in R(Y)$, is the set of all convergent maximal filters of $R(Y)$ containing U . We also note that $k[cl_{E(Y)} k^+[\text{int } U]] = U$.

Received by the editors January 26, 1980 and, in revised form, April 14, 1980.

AMS (MOS) subject classifications (1970). Primary 54D35.

Key words and phrases. Extremally disconnected, P -set, the absolute.

© 1981 American Mathematical Society
0002-9939/81/0000-0178/\$02.25

Let F be any index set and let $2 = \{0, 1\}$ denote the two point discrete space. If J is a finite subset of F and ψ is a function from J into 2 , we will denote $\{f \in 2^F: f \supset \psi\}$ by $[\psi]$. The set $[\psi]$ is a clopen subset of 2^F and the set of all such sets is a base for the topology on 2^F . Recall that 2^F is ccc.

Efimov (Remark 8, p. 274 of [E]) has shown that if X is an ED space, there exists a discrete space D such that βX can be embedded into βD . In Lemma 2.3 we will show that βX can be embedded into βD satisfying a further property which we shall require. A subset P of a space X will be called a P -set of X if P is compact and the intersection of countably many neighborhoods of P is again a neighborhood of P . An equivalent definition is that P is compact and any disjoint F_σ has its closure disjoint from P . We will use the notation " $X \bmod (P)$ " to represent the quotient space of X obtained by collapsing P to a point (even if P is not a P -set).

LEMMA 2.2 (10.50 OF [W]). *A closed irreducible mapping of a Hausdorff space onto an extremally disconnected space is a homeomorphism.* $\square$

LEMMA 2.3. *Let X be an extremally disconnected space and let P be a P -set of βX that is contained in $\beta X - X$. Let D be the discrete space with the same cardinality as X . Then there exist an embedding $h: \beta X \rightarrow \beta D$ and a compact P -set P' of βD such that $P' \cap h[\beta X] = h[P]$.*

PROOF. Let f be any one-to-one map of D onto X . Then βf is a continuous map from βD onto βX . We can choose a compact subset K of βD such that $\beta f|_K$ is a homeomorphism from K onto βX (Lemmas 2.1(2) and 2.2). We claim that $(\beta f)^{-1}[P]$ is a P -set of βD and that $(\beta f)^{-1}[P] \cap K = (\beta f|_K)^{-1}[P]$. Indeed, let $(C_n: n \in \omega)$ be clopen neighborhoods in βD of $(\beta f)^{-1}[P]$. Then $\beta f[\beta D \setminus C_n] \cap P = \emptyset$ for each n . Since P is a P -set of βX , $(\text{cl}_{\beta X}(\bigcup_n \beta f[\beta D \setminus C_n])) \cap P = \emptyset$. Therefore $(\beta f)^{-1}[\text{cl}_{\beta X}(\bigcup_n \beta f[\beta D \setminus C_n])] \cap P = \emptyset$. Therefore $(\beta f)^{-1}[\text{cl}_{\beta X}(\bigcup_n \beta f[\beta D \setminus C_n])] \cap (\beta f)^{-1}[P] = \emptyset$. Also $\beta D \setminus C_n$ is contained in $(\beta f)^{-1}[\beta f[\beta D \setminus C_n]]$. Hence $\text{cl}_{\beta D}(\bigcup_n (\beta D \setminus C_n))$ is contained in $(\beta f)^{-1}[\text{cl}_{\beta X}(\bigcup_n \beta f[\beta D \setminus C_n])]$ and therefore $\bigcap_n C_n$ is a neighborhood of $(\beta f)^{-1}[P]$. Clearly $(\beta f)^{-1}[P] \cap K = (\beta f|_K)^{-1}[P]$. We define h to be $(\beta f|_K)^{-1}$ and P' to be $(\beta f)^{-1}[P]$, and the theorem is proved. $\square$

3. The main result. The crucial step in our main result is Theorem 3.1. We will construct an embedding of a discrete space D into $E(2^F)$, where F is the power set of D . We will do this in such a way that the closure in $E(2^F)$ of our embedded copy of D will be the quotient space of βD with a specified P -set collapsed to a point. As noted in the introduction, our technique was developed by Balcar and Simon [BS]. In the proof of 3.1 we will use the following notation. If D is a discrete space and X is any space with a subset $\{x_d: d \in D\}$, then x is a q -limit of $\{x_d: d \in D\}$, for $q \in \beta D \setminus D$ if $x \in \bigcap_{C \in q} \text{cl}_X\{x_d: d \in C\}$.

THEOREM 3.1. *Let D be any discrete space, let P be a P -set of βD , and let q be any element of $\beta D \setminus D$. Then $\beta D \bmod (P \cup \{q\})$ is homeomorphic to a subspace of $E(2^F)$ where*

$$F = \{C \subseteq D: \text{cl}_{\beta D} C \cap (P \cup \{q\}) = \emptyset\}.$$

PROOF. First we embed D in 2^F . For each $d \in D$, define $x_d \in 2^F$ as follows: for $C \in F$, $x_d(C) = 1$ iff $d \in C$. The set $\{x_d: d \in D\}$ is a faithfully indexed discrete subspace of 2^F . We will assume for the remainder of the proof that $q \notin P$; one can simply ignore all mention of q throughout the proof if $q \in P$.

Fact 1. If $C \subseteq D$ and $(\text{cl}_{\beta D} C) \cap P = \emptyset$ then $\{x_d: d \in C\}$ is C^* -embedded in 2^F . To prove this, let A and B be complementary subsets of C . Without loss of generality $q \notin \text{cl}_{\beta D} A$. Hence $A \in F$. Therefore if we let $\tilde{A} = \{f \in 2^F: f(A) = 1\}$ then $\tilde{A}$ is a clopen subset of 2^F such that $\tilde{A} \cap \{x_d: d \in C\} = \{x_d: d \in A\}$. Therefore C is C^* -embedded in 2^F , and Fact 1 is verified.

Now we choose a subset D_0 of D such that $(\text{cl}_{\beta D} D_0) \cap P = \emptyset$ and $q \in \text{cl}_{\beta D} D_0$. By Fact 1 $\{x_d: d \in D_0\}$ is C^* -embedded in 2^F . Recall that k is the canonical map from $E(2^F)$ onto 2^F . For each $d \in D_0$, choose an element $\mathcal{U}_d$ of $k^{-1}(x_d)$. The set $\{\mathcal{U}_d: d \in D_0\}$ is C^* -embedded in $E(2^F)$; to see this note that k is a continuous map from $\text{cl}_{E(2^F)}\{\mathcal{U}_d: d \in D_0\}$ onto $\text{cl}_{2^F}\{x_d: d \in D_0\}$. The fact that $\{x_d: d \in D_0\}$ is discrete and C^* -embedded in 2^F , together with Lemma 2.2, implies that $\{\mathcal{U}_d: d \in D_0\}$ is C^* -embedded in $E(2^F)$. We can therefore find $\mathcal{U} \in \text{cl}_{E(2^F)}\{\mathcal{U}_d: d \in D_0\}$ such that $\{\mathcal{U}\} = \bigcap_{A \in q} \text{cl}_{E(2^F)}\{\mathcal{U}_d: d \in A\}$. If $C \in F$ then $D \setminus C \in q$ since $\text{cl}_{\beta D} C \cap (P \cup \{q\}) = \emptyset$. This implies that for each $C \in F$, $\{f \in 2^F: f(C) = 0\}$ contains $k(\mathcal{U})$ since $\{k(\mathcal{U})\} = \bigcap_{A \in q} \text{cl}_{2^F}\{x_d: d \in A\}$ and $x_d(C) = 1$ iff $d \in C$. Therefore $k(\mathcal{U})$ is the function from F to 2 which is identically 0; denote it by $\underline{0}$. To complete the embedding of D in $E(2^F)$, we will now choose, for each $d \in D \setminus D_0$, an ultrafilter $\mathcal{U}_d \in k^{-1}(x_d)$. First we need some notation. Because, as noted earlier, 2^F satisfies the countable chain condition, for each $U \in \mathcal{U}$, there is a countable family of clopen sets $[\Psi_n]$ such that $\bigcup_n [\Psi_n]$ is a dense subset of U . We will let $s(U) = \bigcup (\text{dom } \Psi_n)$. For each n , $\text{dom } \Psi_n$ is a finite subset of F ; hence $s(U)$ is a countable subset of F . For each $d \in D \setminus D_0$ we will let $\mathcal{F}_d = \{U \in \mathcal{U}: C \in s(U) \text{ implies } x_d(C) = 0\}$.

Fact 2. If $U_1, U_2, \dots, U_n$ are all in $\mathcal{F}_d$ for some $d \in D \setminus D_0$ then $x_d \in \text{cl}_{2^F} \text{int}_{2^F} \bigcap_{i=1}^n U_i$.

To prove Fact 2 we first observe that, by the definition of $\mathcal{F}_d$, for each $C \in \bigcup_{i=1}^n s(U_i)$, $x_d(C) = 0$. Let $[\Psi]$ be a basic neighborhood of x_d and let $\text{dom } \Psi = J_1 \cup J_2$ where $J_1 \cap \bigcup_{i=1}^n s(U_i) = \emptyset$ and $J_2 \subset \bigcup_{i=1}^n s(U_i)$. Therefore $[\Psi|_{J_2}]$ is a neighborhood of $\underline{0}$. Notice, also, that $U_i \in \mathcal{U}$ for $i = 1, 2, \dots, n$ and therefore $\underline{0} \in \text{cl}_{2^F} \text{int}_{2^F} \bigcap_{i=1}^n U_i$. Hence we can find basic clopen sets $[\Psi_i]$ for $i = 1, \dots, n$ with $[\Psi_i] \subseteq U_i$, $\text{dom } \Psi_i \subseteq s(U_i)$ and $[\Psi|_{J_2}] \cap [\Psi_1] \cap \dots \cap [\Psi_n] \neq \emptyset$. Let f be an element of $[\Psi|_{J_2}] \cap [\Psi_1] \cap \dots \cap [\Psi_n]$ and define $\hat{f}(C) = f(C)$ for $C \notin J_1$ and let $\hat{f}(C) = \Psi(C)$ for $C \in J_1$. Since $(\text{dom } \Psi_i) \cap J_1 = \emptyset$ for each i we conclude that $\hat{f} \in [\Psi] \cap [\Psi_1] \cap \dots \cap [\Psi_n]$. Hence $x_d \in \text{cl}_{2^F} \text{int}_{2^F} \bigcap_{i=1}^n U_i$ and Fact 2 is verified.

It follows from Fact 2 that $\mathcal{F}_d$ is a filter base in $R(2^F)$ and that x_d is an accumulation point of $\mathcal{F}_d$. Therefore using a Zorn's lemma argument we can choose an ultrafilter $\mathcal{U}_d$ such that $\mathcal{U}_d$ contains $\mathcal{F}_d$ and $k(\mathcal{U}_d) = x_d$.

Fact 3. For any $U \in \mathcal{U}$ there is an $A \subseteq D$ such that $\text{cl}_{\beta D} A \supset P$ and $\{\mathcal{V} \in E(2^F): U \in \mathcal{V}\} \supset \{\mathcal{U}_d: d \in A\}$.

To prove Fact 3, let U be an element of $\mathcal{U}$ and list $s(U)$ as $\{C_n: n \in \omega\}$. Each C_n is in F ; hence $(\text{cl}_{\beta D} C_n) \cap P = \emptyset$. Let $A = D \setminus (\bigcup_n C_n \cup D_0)$; then $\text{cl}_{\beta D} A$ contains P , since P is a P -set of βD and $\text{cl}_{\beta D} D_0 \cap P = \emptyset$. For each $d \in A$ and $n \in \omega$, $x_d(C_n) = 0$ which means that $U \in \mathcal{F}_d$. Therefore $U \in \mathcal{U}_d$ and $\mathcal{U}_d \in \{\mathcal{V} \in E(2^F): U \in \mathcal{V}\}$ and Fact 3 is verified.

To complete the proof of the theorem, we will now show that $\text{cl}_{E(2^F)}\{\mathcal{U}_d: d \in D\}$ is homeomorphic to $\beta D \bmod (P \cup \{q\})$.

Fact 4. $\text{cl}_{E(2^F)}\{\mathcal{U}_d: d \in D\}$ is homeomorphic to $\beta D \bmod (P \cup \{q\})$. First we will show that $\text{cl}_{2^F}\{x_d: d \in D\}$ is homeomorphic to $\beta D \bmod (P \cup \{q\})$. Let g denote the map which takes each $d \in D$ to $x_d \in 2^F$. Then βg is a map from βD onto $\text{cl}_{2^F}\{x_d: d \in D\}$. We have already seen that $\underline{0}$ is the q -limit of $\{x_d: d \in D\}$; one can similarly show that, for any $p \in P$, $\underline{0}$ is the p -limit of $\{x_d: d \in D\}$ as well. Also, for any $t \in \beta D \setminus (P \cup \{q\})$, $\underline{0}$ is not the t -limit of $\{x_d: d \in D\}$. To see this, note that for any such t there exists a $C \in t$ such that $\text{cl}_{\beta D} C \cap (P \cup \{q\}) = \emptyset$ and thus $\{f \in 2^F: f(C) = 1\}$ is a neighborhood of the t -limit of $\{x_d: d \in D\}$ which does not contain $\underline{0}$. These facts together with Fact 1 imply that βg is a map from βD onto $\text{cl}_{2^F}\{x_d: d \in D\}$ such that $\beta g[P \cup \{q\}] = \underline{0}$ and otherwise βg is one-to-one.

Now let us consider the map k restricted to $\text{cl}_{E(2^F)}\{\mathcal{U}_d: d \in D\}$. To complete the proof of Fact 4 it suffices to show that the preimage of $\underline{0}$ under $k|_{\text{cl}_{E(2^F)}\{\mathcal{U}_d: d \in D\}}$ is a singleton. Recall that $\mathcal{U}$ was chosen in $\text{cl}_{E(2^F)}\{\mathcal{U}_d: d \in D\}$ so that $k(\mathcal{U}) = \underline{0}$. Suppose that $\mathcal{V} \in \text{cl}_{E(2^F)}\{\mathcal{U}_d: d \in D\}$ and $\mathcal{V} \neq \mathcal{U}$. We can therefore choose $U \in \mathcal{U}$ such that $U \notin \mathcal{V}$. By Fact 3 there is an $A \subset D$ such that $\text{cl}_{\beta D} A \supset P$ and $\{\mathcal{U} \in E(2^F): U \in \mathcal{U}\} \supset \{\mathcal{U}_d: d \in A\}$. Also since $\mathcal{U}$ is the q -limit of $\{\mathcal{U}_d: d \in D\}$ and $\{\mathcal{U} \in E(2^F): U \in \mathcal{U}\}$ is a clopen neighborhood of $\mathcal{U}$ we can assume $A \in q$. Therefore $k(\mathcal{V})$ is an element of $\text{cl}_{2^F}\{x_d: d \in D \setminus A\}$ and since $D \setminus A \in F$ we observe that $k(\mathcal{V}) \neq \underline{0}$. Therefore $\beta D \bmod (P \cup \{q\})$ is embedded in $E(2^F)$. This verifies Fact 4, and concludes the proof of the theorem. $\square$

We can now prove the theorem stated in the introduction which is our main result.

THEOREM 3.2. *Let X be an ED space which is not weakly Lindelöf nor almost compact. Then there is a P -set, P , of βX and a point $q \in \beta X \setminus X$ such that $|P \cup \{q\}| > 1$. Also $\beta X \bmod (P \cup \{q\})$ can be embedded into $E(2^\lambda)$ where $\lambda = 2^{|X|}$.*

PROOF. Since X is not weakly Lindelöf there is a cover $\mathcal{C}$ of X consisting of clopen sets so that no countable subcollection of $\mathcal{C}$ has dense union. Define $P \subset \beta X$ to be $\bigcap \{\beta X \setminus \text{cl}_{\beta X}[\bigcup \mathcal{E}]: \mathcal{E} \text{ is a countable subcollection of } \mathcal{C}\}$. One can check that P is a P -set of βX and that $P \subseteq \beta X \setminus X$. Since X is not almost compact, $|\beta X \setminus X| > 1$ [GJ, 6J]; hence a point $q \in \beta X \setminus X$ can be chosen to ensure that $|P \cup \{q\}| > 1$. By Lemma 2.3 we can embed βX into βD , with $|D| = |X|$, such that P is the restriction to βX of a P -set, P' , of βD . We now observe that by 3.1 we can regard $\beta D \bmod (P' \cup \{q\})$ as being a subspace of $E(2^F)$ where $|F| = 2^{|D|}$. This completes the proof. $\square$

THEOREM 3.3. *For an ED space X , the following are equivalent:*

- (a) *X is weakly Lindelöf or almost compact,*
- (b) *X is C^* -embedded in each ED space containing it,*
- (c) *X is C^* -embedded in each ED space containing it as a closed subspace.*

PROOF. (a) implies (b) and (b) implies (c) are obvious. For (c) implies (a) let X be an ED space which is not weakly Lindelöf nor almost compact. Let P and q be as in 3.2, then $\beta X \bmod (P \cup \{q\})$ can be embedded in the ED space $E(2^\lambda)$ for an appropriate λ . Clearly $Y = E(2^\lambda) \setminus [\text{cl } X \setminus X]$ is a dense subset of $E(2^\lambda)$ and is therefore a C^* -embedded ED subspace of $E(2^\lambda)$ by 2.1. Thus X is a closed subset of Y but is not C^* -embedded because X is not C^* -embedded in $\text{cl}_{E(2^\lambda)} X = \beta X \bmod (P \cup \{q\})$. $\square$

We remark in connection with Theorem 3.3 that an almost compact ED space of nonmeasurable cardinality is weakly Lindelöf. To see this we note that an extremally disconnected space X which is not weakly Lindelöf has a P -set in its growth. If in addition, X is almost compact, this P -set is a singleton, hence a P -point. Isbell has shown that an extremally disconnected space of nonmeasurable cardinality has no nonisolated P -points (12H of [GJ], [II]).

REFERENCES

- [A] C. E. Aull, *Absolute C^* -embedding of P -spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. **26** (1978), 831–836.
- [BS] B. Balcar and P. Simon, *Convergent nets in the spaces of uniform ultrafilters*, (preprint) 1978.
- [CHN] W. W. Comfort, N. Hindman and S. Negrepontis, *F' -spaces and their product with P -spaces*, Pacific J. Math. **28** (1969), 489–502.
- [E] B. A. Efimov, *Extremally disconnected spaces and absolutes*, Trans. Moscow Math. Soc. **23** (1970), 243–285.
- [GJ] L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N. J., 1960.
- [II] J. R. Isbell, *Zero-dimensional spaces*, Tôhoku Math. J. **7** (1955), 7–8.
- [W] R. C. Walker, *The Stone-Cech compactification*, Springer-Verlag, Berlin and New York, 1974.
- [Wo] R. G. Woods, *A survey of absolutes of topological spaces*, Top. Structures **2**, Math. Centre Tracts, no. 116, 1979, pp. 323–362.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MANITOBA, WINNIPEG, MANITOBA, CANADA R3T 2N2

Current address: Department of Mathematics, University of Kansas, Lawrence, Kansas 66045