SOME PRODUCTS OF TOPOLOGICAL SPACES WHICH ARE MANIFOLDS

CHARLES D. BASS

ABSTRACT. We give some conditions which imply that a product $X \times Y$ of two metric spaces has the disjoint disks property. As a consequence the products of certain cell-like images of manifolds are shown to be manifolds.

1. Introduction. Recent work of Cannon and Edwards, [4], [7], together with unpublished results of F. Quinn have led to simply stated topological criteria for identifying n -manifolds. One of these, of particular interest in this paper, is the celebrated "disjoint disks property". A space X is said to have this property if any two maps f_1 and f_2 from a 2-cell D into X can be approximated by maps f'_1 and f'_2 respectively from D into X such that $f'_1(D) \cap f'_2(D) = \emptyset$. Other authors [1], [3], [8] using the techniques of decomposition space theory have shown that some manifolds may be realized as the product of two spaces X and Y which fail to be manifolds. In this note we will show that certain product spaces have the disjoint disks property, thereby identifying some unusual factorizations of manifolds into nonmanifold factors. This result is Theorem 2 stated in §3.

2. Notation. Let X be a metric space, ρ a metric on X . For a point $p \in X$ and a number $\epsilon > 0$, the ϵ -neighborhood of p is $N(p, \epsilon) = \{q | \rho(p, q) < \epsilon\}$. If $A \subset X$ the diameter of A is denoted by $\text{diam } A$; $\text{diam } A = \sup_{x, y \in A} \rho(x, y)$. If Q is a compact space and f_1 and f_2 are maps of Q into a metric space X , we define the distance between f_1 and f_2 by $\rho(f_1, f_2) = \sup_{x \in Q} \rho(f_1(x), f_2(x))$.

If M is an n -manifold the terms $\text{Bd } M$ and $\text{Int } M$ represent the sets of boundary and interior points respectively. The unit n -cell in Euclidean n -space is denoted B^n ; the unit $(n-1)$ -sphere $\text{Bd } B^n = S^{n-1}$. Let X be a metric space, $A \subset X$, $p \in \text{cl}(X - A)$. Then $X - A$ is k -LC at p if for every $\epsilon > 0$ there is some $\delta > 0$ such that each map $f: S^k \rightarrow N(p, \delta) - A$ extends to a map $g: B^{k+1} \rightarrow N(p, \epsilon) - A$. Often X is said to be *locally simply connected* if X is 1-LC at each point $p \in X$.

Finally, suppose that X and Y are metric spaces endowed with metrics ρ_1 and ρ_2 respectively. Then a product metric ρ on $X \times Y$ is induced by ρ_1 and ρ_2 according to the rule $\rho((x_1, y_1), (x_2, y_2)) = \max\{\rho_1(x_1, x_2), \rho_2(y_1, y_2)\}$. We will refer to the canonical projection maps from $X \times Y$ onto X and Y respectively as π_1 and π_2 .

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3. Mapping a disk into product spaces. There are two essential steps in our technique. First we obtain mappings of the unit 2-cell B^2 into a product $X \times Y$ having relatively simple intersections with certain subspaces of the form $J_i \times Y$, J_i a simple closed curve in X . Then we adjust mappings of B^2 into $J_i \times Y$ using the following lemma.

LEMMA 1. *Let Y be a locally compact, locally simply connected metric space such that no 1-dimensional subset of Y contains an open subset of Y and, for each arc A in Y , $Y - A$ is 0-LC at each point of A . Let $A_1, \dots, A_k$ be arcs in S^1 , $x_1, \dots, x_k$ distinct points in Y , ϵ a positive number, K a finite 2-complex, and f a map of K into $S^1 \times Y$. Then there exists a map $f': K \rightarrow S^1 \times Y$ such that*

(1) $\rho(\pi_2 f, \pi_2 f') < \epsilon$, and

(2) $f'(K) \cap (\cup_{i=1}^k (A_i \times \{x_i\})) = \emptyset$.

Moreover, if C is a closed subset of $K - f^{-1}\pi_2^{-1}(\{x_0, \dots, x_k\})$, then f' can be obtained satisfying

(3) $f'|_C = f|_C$.

PROOF. We content ourselves with a sketch since the details are relatively routine. First we triangulate K by a triangulation T with fine mesh so that the 1-skeleton $T^{(1)}$ may be regarded as the union of simple closed curves which map under f into small open sets in $S^1 \times Y$. In fact, we may suppose that the image under f of each 2-simplex of T intersects at most one of the arcs $A_i \times \{x_i\}$ and any such image intersecting $f(C)$ misses all of the arcs $A_i \times \{x_i\}$. Next, we approximate f by a map f_1 such that $\pi_2 f_1(T^{(1)}) \subset Y - \{x_1, \dots, x_k\}$. Then, altering only the S^1 -coordinates of points of $f_1(K)$, we replace the images of 2-simplexes near the arcs $A_i \times \{x_i\}$ in the following manner.

For each $\sigma_j \in T^{(2)}$ such that $f_1(\sigma_j) \cap (A_i \times \{x_i\}) \neq \emptyset$ for some i , let N be an annular neighborhood of $\text{Bd } \sigma_j$ in σ_j such that $\pi_2 f_1(N) \subset Y - \{x_1, \dots, x_k\}$. Let J denote the boundary component of N in the interior of σ_j . Using a contraction of $\pi_1 f_1(\text{Bd } \sigma_j)$ in S^1 we obtain a map $g: N \rightarrow S^1$ such that $g(J)$ is a point in $S^1 - A_i$ and $g|_{\text{Bd } \sigma_j} = \pi_1 f_1|_{\text{Bd } \sigma_j}$. Then we define f' on σ_j by the rule

$$f'(x) = \begin{cases} (g(x), \pi_2 f_1(x)) & \text{if } x \in N, \\ (g(J), \pi_2 f_1(x)) & \text{if } x \in \sigma_j - N. \end{cases}$$

Property (1) depends only on restrictions imposed on f_1 since $\pi_2 f' = \pi_2 f_1$. Property (2) is a consequence of three facts: that $\pi_2 f_1(\sigma_j)$ contains at most one point of $\{x_1, \dots, x_k\}$ for each σ_j ; that $\pi_2 f_1(N) \subset Y - \{x_1, \dots, x_k\}$; and that $g(J) \subset S^1 - A_i$. Property (3) holds because $f' = f$ on all simplexes of K which intersect C .

THEOREM 2. *Let X and Y be locally compact, locally simply connected metric spaces such that no 1-dimensional subset of either space contains an open subset and, for each arc in X (or Y), $X - A$ ($Y - A$) is 0-LC at each point of A . Then $X \times Y$ has the disjoint disks property.*

PROOF. Let ρ_1, ρ_2 be metrics on X and Y respectively; ρ the product metric on $X \times Y$ induced by ρ_1 and ρ_2 . Let f_1 and f_2 be maps from B^2 into $X \times Y$, ϵ a

positive number. A first objective is to obtain approximations f'_1 and f'_2 of f_1 and f_2 respectively such that $f'_1(B^2) \cap f'_2(B^2)$ lies in the union of finitely many 1-dimensional continua in distinct levels $X \times \{y_i\}$.

By the local homotopy assumptions on X and Y , there exist for each $i = 1, 2$ maps $\varphi_i: B^2 \rightarrow X$ and $\theta_i: B^2 \rightarrow Y$ and a triangulation T_0 of B^2 such that

- (1) $\text{diam } f_i(\sigma_j) < \varepsilon/9$ for each simplex σ_j of T_0 ,
- (2) $\rho(\varphi_i, \pi_1 f_i) < \varepsilon/9$,
- (3) $\rho(\theta_i, \pi_2 f_i) < \varepsilon/3$,
- (4) $\varphi_i|T_0^{(1)}$ and $\theta_i|T_0^{(1)}$ are embeddings, and
- (5) $\varphi_1(T_0^{(1)}) \cap \varphi_2(T_0^{(1)}) = \emptyset$.

(Here we are using $T_0^{(1)}$ to denote the space underlying $T_0^{(1)}$ as well as the simplicial complex consisting of all 1-simplexes and vertices of T_0 .)

For each 2-simplex $\sigma_j \in T_0$ choose $y_{1,j} \in \theta_1(\sigma_j)$, $y_{2,j} \in \theta_2(\sigma_j)$. These points must be chosen so that $y_{i,j} \neq y_{p,q}$ if either $i \neq p$ or $j \neq q$. Let F denote the set of such points $y_{i,j}$. For each $i = 1, 2$ there exist homotopies $H_{i,j}: \text{Bd } \sigma_j \times [0, 1] \rightarrow \theta_i(\sigma_j)$ such that $H_{i,j}(p, 0) = \theta_i(p)$ and $H_{i,j}(p, 1) = y_{i,j}$ for each $p \in \text{Bd } \sigma_j$. Now, choose a neighborhood N_j of $\text{Bd } \sigma_j$ in σ_j homeomorphic to $\text{Bd } \sigma_j \times [0, 1]$. Let α_j be a homeomorphism from N_j onto $\text{Bd } \sigma_j \times [0, 1]$ which sends each s in $\text{Bd } \sigma_j$ to $(s, 0)$ in $\text{Bd } \sigma_j \times [0, 1]$; let β_j be a map from σ_j onto σ_j such that $\beta_j|N_j$ is a retraction onto $\text{Bd } \sigma_j$ and $\beta_j|\text{cl}(\sigma_j - N_j)$ is a homeomorphism. For each $i = 1, 2$ and approximation f'_i is defined simplex by simplex by the rule

$$(f'_i|_{\sigma_j})(p) = \begin{cases} (\varphi_i \beta_j(p), H_{i,j} \alpha_j(p)) & \text{if } p \in N_j, \\ (\varphi_i \beta_j(p), y_{i,j}) & \text{if } p \in \sigma_j - N_j. \end{cases}$$

Each map f'_i defined in this way satisfies the following:

- (6) for each 2-simplex σ_j in T_0 , $f'_1(\sigma_j) \cap f'_2(B^2) \subset (\varphi_2(T_0^{(1)}) \times \{y_{1,j}\}) \cup [\cup_k (\varphi_1(\text{Bd } \sigma_j) \times \{y_{2,k}\})]$;
- (7) for each 2-simplex σ_j in T_0 , $f'_2(\sigma_j) \cap f'_1(B^2) \subset (\varphi_1(T_0^{(1)}) \times \{y_{2,j}\}) \cup [\cup_k (\varphi_2(\text{Bd } \sigma_j) \times \{y_{1,k}\})]$;
- (8) $\pi_1 f'_i(\sigma_j) = \varphi_i(\sigma_j)$ for each i ;
- (9) $\pi_2 f'_i(\sigma_j) = \theta_i(\sigma_j)$ for each i ;
- (10) $\pi_1 f'_i|T_0^{(1)} = \varphi_i|T_0^{(1)}$ for each i ; and
- (11) $\pi_2 f'_i|T_0^{(1)} = \theta_i|T_0^{(1)}$ for each i .

Our next step is to modify each f'_i slightly in the factor Y , obtaining maps f''_1 and f''_2 from B^2 into $X \times Y$ such that, for each 2-simplex σ_j in T_0 ,

$$[f''_1(B^2) \cap (\varphi_2(\text{Bd } \sigma_j) \times Y)] \cup [f''_2(B^2) \cap (\varphi_1(\text{Bd } \sigma_j) \times Y)]$$

lies within the finite union of arcs in distinct levels $[\varphi_1(\text{Bd } \sigma_j) \cup \varphi_2(\text{Bd } \sigma_j)] \times \{z_k\}$. To this end choose triangulations $T_1, T_2, T_3, \dots$ such that T_i subdivides T_{i-1} for each i and $\lim_{i \rightarrow \infty} \{\text{mesh } T_i\} = 0$. Again, by the use of the local homotopy assumption on X , approximate each φ_i by a map ψ_i , $i = 1, 2$, such that

- (12) $\rho_1(\psi_i, \varphi_i) < \varepsilon/3$,
- (13) $\psi_1(T_j^{(1)}) \cap \varphi_2(T_0^{(1)}) = \emptyset$ for each $j = 1, 2, 3, \dots$,
- (14) $\psi_2(T_j^{(1)}) \cap \varphi_1(T_0^{(1)}) = \emptyset$ for each j , and
- (15) $\psi_i|T_0^{(1)} = \varphi_i|T_0^{(1)}$.

Choose $\delta > 0$ such that no subset of $X \times Y$ having diameter less than δ contains a simple closed curve of the form $\varphi_i(\text{Bd } \sigma_j) \times \{y\}$, $\sigma_j \in T_0^{(2)}$, $y \in Y$. Let m be a positive integer such that

(16) $\text{diam } \psi_i(\tau_k) < \delta$ for each simplex τ_k of T_m , $i = 1, 2$.

For each 2-simplex $\sigma_j \in T_0$ let $U_{1,j}$ and $U_{2,j}$ be path-connected open neighborhoods of

$$y_{1,j} = \pi_2(f'_1(\sigma_j) - (\varphi_1(\text{Bd } \sigma_j) \times Y)) \quad \text{and} \quad y_{2,j} = \pi_2(f'_2(\sigma_j) - (\varphi_2(\text{Bd } \sigma_j) \times Y))$$

respectively such that

(17) $\text{diam } U_{i,j} < \varepsilon/3$.

Assume in addition that $U_{i,j} \cap U_{p,q} = \emptyset$ unless $i = p$, $j = q$. Let $\tau_{j,1}, \dots, \tau_{j,s(j)}$ denote the 2-simplexes of T_m lying in σ_j . Let $z_{1,j,1}, \dots, z_{1,j,s(j)}$ be distinct points of $U_{1,j}$; let $z_{2,j,1}, \dots, z_{2,j,s(j)}$ be distinct points of $U_{2,j}$ missing $\theta_1(T_0^{(1)}) \cup \theta_2(T_0^{(1)}) \cup F$. As the choice of σ_j varies, the total collection F' of points $z_{i,j,k}$ engendered by the subdivision T_m must have the property $z_{i,j,k} \neq z_{p,q,r}$ whenever $i \neq p$, $j \neq q$, or $k \neq r$. Since $U_{1,j}$ and $U_{2,j}$ are path-connected there exists for each i, j , and $k = 1, \dots, s(j)$ a homotopy $H_{i,j,k}: \text{Bd } \tau_{j,k} \times [0, 1] \rightarrow U_{i,j}$ such that $H_{i,j,k}(\text{Bd } \tau_k \times \{0\}) = y_{i,j}$ and $H_{i,j,k}(\text{Bd } \tau_k \times \{1\}) = z_{i,j,k}$. By making use of the technique used to define f'_1 and f'_2 we obtain maps $\mu_{1,j}$ and $\mu_{2,j}$ from each 2-simplex σ_j of T_0 into $X \times Y$ such that

(18) for each $\tau_{j,k} \in T_m^{(2)} \cap \sigma_j$, $\mu_{i,j}(\tau_{j,k}) \subset [\psi_i(\text{Bd } \tau_{j,k}) \times U_{i,j}] \cup [\psi_i(\tau_{j,k}) \times \{z_{i,j,k}\}]$, and

(19) $\pi_1 \mu_{i,j}|(T_m^{(1)} \cap \sigma_j) = \psi_i|(T_m^{(1)} \cap \sigma_j)$.

The promised approximations f''_i are defined simplex by simplex by the rule

$$f''_i(p) = \begin{cases} f'_i(p) & \text{if } p \in N_j, \\ \mu_{i,j}\beta_j(p) & \text{if } p \in \sigma_j - N_j. \end{cases}$$

Since $z_{1,j,k} \neq z_{2,p,q}$ for all values j, k, p , and q , property (18) implies that $[f''_1(B^2) - (\varphi_1(T_0^{(1)}) \times Y)] \cap [f''_2(B^2) - (\varphi_2(T_0^{(1)}) \times Y)] = \emptyset$. Thus, by (13), (16), and (18), for each $\sigma_j \in T_0^{(2)}$, $f''_1(B^2) \cap (\varphi_2(\text{Bd } \sigma_j) \times Y)$ is confined to the union of finitely many arcs $A_{1,j,k} \times \{z_{1,j,k}\}$, $A_{1,j,k} \subset \varphi_2(\text{Bd } \sigma_j)$. Similarly, by (14), (16), and (18), $f''_2(B^2) \cap (\varphi_1(\text{Bd } \sigma_j) \times Y)$ is confined to the union of finitely many arcs $A_{2,j,k} \times \{z_{2,j,k}\}$, $A_{2,j,k} \subset \varphi_1(\text{Bd } \sigma_j)$.

We complete the proof by applying Lemma 1 to approximate each map $f''_i|N_j$ by a map $\chi_{i,j}: N_j \rightarrow \varphi_i(\text{Bd } \sigma_j) \times Y$ such that

(20) $\rho(\pi_2 \chi_{i,j}, \pi_2 f''_i|N_j) < \varepsilon/3$,

(21) $\chi_{1,j}(N_j) \cap f''_2(B^2) = \emptyset$,

(22) $\chi_{2,j}(N_j) \cap f''_1(B^2) = \emptyset$, and

(23) $\chi_{i,j}|\text{Bd } N_j = f''_i|\text{Bd } N_j = f'_i|\text{Bd } N_j$.

Using these maps we define two maps $f'''_i: B^2 \rightarrow X \times Y$ ($i = 1, 2$) by putting $f'''_i = \chi_{i,j}$ on each annulus N_j and by putting $f'''_i = f''_i$ on $B^2 - (\cup_j N_j)$. Properties (21), (22), and (23) clearly imply that $f'''_1(B^2) \cap f'''_2(B^2) = \emptyset$. It remains only to check that $\rho(f_i, f'''_i) < \varepsilon$.

Suppose that $p \in N_j$ for some j . Then, conditions (1) and (2) imply that

$\text{diam } \varphi_i(\text{Bd } \sigma_j) < \varepsilon/3$. This fact combined with (20) implies that $\rho(f_i'''(p), f_i''(p)) < \varepsilon/3$. Thus $\rho(f_i'''(p), f_i(p)) < \varepsilon/3 + \rho(f_i''(p), f_i(p))$. Moreover, $f_i''(p) = f_i'(p)$ and $\pi_1 f_i'(p) = \varphi_1(q)$ for some $q \in \sigma_j$ by (8). Thus condition (1) and (2) yield

$$\rho(\pi_1 f_i''(p), \pi_1 f_i(p)) < \varepsilon/3 + \varepsilon/3.$$

By (9) we have $\pi_2 f_i'(p) = \theta_i(q')$ for some $q' \in \sigma_j$. Conditions (1) and (3) yield

$$\rho(\pi_2 f_i''(p), \pi_2 f_i(p)) < \varepsilon/3 + \varepsilon/3.$$

Combining these we have $\rho(f_i'''(p), f_i(p)) < \varepsilon$.

Now, suppose that $p \in \sigma_j - N_j$ for some j . Thus, $f_i'''(p) = f_i''(p)$. Property (18) implies that $\pi_1 f_i''(p) = \psi_i(q)$ for some $q \in \sigma_j$. Conditions (1), (2), and (12) yield

$$\rho(\psi_i(q), \pi_1 f_i(p)) < \varepsilon.$$

Conditions (17) and (18) yield

$$\rho(\pi_2 f_i''(p), \pi_2 f_i(p)) < \varepsilon/3 + \rho(\pi_2 f_i'(p), \pi_2 f_i(p)).$$

By (9), $\pi_2 f_i'(p) = \theta_i(q')$ for some $q' \in \sigma_j$. Thus (1) and (3) yield

$$\rho(\pi_2 f_i'(p), \pi_2 f_i(p)) < \varepsilon/3 + \varepsilon/3.$$

Combining the above gives $\rho(f_i'''(p), f_i(p)) < \varepsilon$ as required. This completes the proof of Theorem 2.

4. Products of cell-like images of manifolds. The following result gives abundant examples of factorizations of certain manifolds into nonmanifold factors.

COROLLARY 3. *Let M, N be finite-dimensional manifolds of dimension at least three without boundary. Let P_1 and P_2 be proper cell-like maps from M and N respectively onto topological spaces X and Y . If X and Y are ANR's then $X \times Y$ is homeomorphic to $M \times N$.*

PROOF. Since P_1 and P_2 are proper cell-like maps the product $P_1 \times P_2: M \times N \rightarrow X \times Y$ is a proper cell-like map as well. Since X and Y are ANR's, $X \times Y$ is an ANR. Furthermore, X and Y are generalized manifolds by [9]. It follows from Lemma 2.1 and Proposition 2.2 of [6] that X and Y satisfy the hypotheses of Theorem 2. Thus, Edwards' approximation theorem [7] implies that $P_1 \times P_2$ is approximable by homeomorphisms.

It is presently unknown whether the prior result remains true in case $\dim N = 1$. If $\dim M = \dim N = 2$, the result is classical. In case $\dim M > 3$, $\dim N = 2$, the result is due to R. J. Daverman.

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DEPARTMENT OF MATHEMATICS, PEMBROKE STATE UNIVERSITY, PEMBROKE, NORTH CAROLINA 28372