PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 4, April 1981

GENERALIZATIONS OF CERTAIN FUNDAMENTAL RESULTS ON FINITE GROUPS

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In this short note, we synthesize various fundamental results of finite groups and their proofs to obtain a generalization of J. G. Thompson's fundamental $A \times B$ Lemma (cf. [5, Theorem 5.3.4]), a generalization of a fundamental result of H. Bender (cf. [2, Satz] and [4, Theorem 3.1]) and a generalization of an important result of N. Blackburn (cf. [3, Theorem]).

Our notation is standard and tends to follow the notation of [5]. In particular, p denotes a prime integer and all groups that we consider are finite.

Our first four results concern the following basic situation.

V is a p-group acted on by the group G where $G = KP$ with $K = O_{p'}(G)$ and $P \in \mathrm{Syl}_p(G)$.

We begin by observing that the proof of [2, Satz] yields a generalization of [5, Theorem 5.2.4].

THEOREM 1. *Suppose that V is abelian. Let $k \in K$ be such that $\Omega_1(C_V(P)) \leq C_V(k)$. Then $[V, k] = 1$.*

PROOF. By [5, Theorem 5.2.4] and the observation that $\Omega_1(C_V(P)) = C_{\Omega_1(V)}(P)$, we may assume that $V = \Omega_1(V)$. Moreover, by replacing G by $\bar{G} = G/C_K(V)$, we may assume that $C_K(V) = 1$. Also $K_1 = C_K(C_V(P))$ is P-invariant and hence we may assume that $K = K_1$ and $C_V(P) \leq C_V(G)$. But $V = C_V(K) \times [V, K]$ and P acts on $[V, K]$ by [5, Theorem 5.2.3]. Suppose that $[V, K] \neq 1$. Then $1 \neq C_V(P) \cap [V, K] \leq C_V(K)$ which is a contradiction. Thus $[V, K] = 1$ and we are done.

The next result also applies the proof of [2, Satz] to generalize the fundamental results [5, Theorems 5.3.4 and 5.3.10] and [4, Theorem 2.4] when $p \neq 2$.

THEOREM 2. *Suppose that $p \neq 2$. Let $k \in K$ be such that $\Omega_1(C_V(P)) \leq C_V(k)$. Then $[V, k] = 1$.*

PROOF. As above, we may assume that $C_K(V) = 1$ and $\Omega_1(C_V(P)) \leq C_V(G)$. Then [5, Theorem 5.3.13] implies that V contains a characteristic subgroup D of class at most 2 and exponent p such that K acts faithfully on D. Then $C_D(P) \leq \Omega_1(C_V(P)) \leq C_V(G)$ and hence we may assume that $V = D$. As in [2, (B)], we now use an observation of R. Baer (cf. [1, Theorem B.1]) to conclude that we may assume that V is abelian. But then $C_V(P) \leq C_V(G)$ and an application of Theorem 1 completes the proof.

Received by the editors July 9, 1980 and, in revised form, August 21, 1980.
1980 *Mathematics Subject Classification*. Primary 20D45; Secondary 20E34.

0002-9939/81/0000-0182/$01.75

Note that Theorem 2 is false for $p = 2$. We give two examples.

(1) Let V be a direct product of $n \geq 1$ quaternion groups of order 8 and let $G = KP$ where K is an elementary abelian group of order 3^n, $P^{\#}$ consists of an involution inverting every element of K and where $C_V(P) = C_V(K) = \Omega_1(V)$.

(2) Let V be a 2-group of type $U_3(4)$ and let $G = KP \leq \mathrm{Aut}(V)$ with G a Frobenius group of order 20 such that $C_V(P) \leq C_V(K) = Z(V) = \Omega_1(V)$, (cf. [**6**, VI, Lemma 2.5]).

However, utilizing the proof of [**4**, Theorem 3.1], we can demonstrate

THEOREM 3. *Suppose that $p = 2$. Let $k \in K$ be such that $\langle [V, P], \Omega_2(C_V(P))\rangle \leq C_V(k)$. Then $[V, k] = 1$.*

PROOF. As above, we may assume that $C_K(V) = 1$, $\langle [V, P], \Omega_2(C_V(P))\rangle \leq C_V(K)$ and $\Omega_2(C_V(P)) \leq C_V(G)$. Also, since G acts on $C_V(K)$ and on $N_V(C_V(K))$, we may assume that $C_V(K) \trianglelefteq V$. As $V = [V, K]C_V(K)$ by [**5**, Theorem 5.3.5], $[V, K] = [V, K, K]$ by [**5**, Theorem 5.3.6] and $[V, K]$ is G-invariant by [**5**, Theorem 2.2.1(iii)], we may assume that $V = [V, K]$. Theorem 1 implies that K acts trivially on every characteristic abelian subgroup of V. Hence [**5**, Exercise 5.4] implies that V is a nonabelian special 2-group. Thus $\exp(V) = 4$ and $C_V(P) \leq C_V(G)$. On the other hand, $K = [K, P]C_K(P)$ and $P \times C_K(P)$ acts on V. Since $C_K(V) = 1$, [**5**, Theorem 5.3.4] implies that $C_K(P) = 1$ and $K = [K, P]$. Note that $[V, P, K] = 1$ and $[K, V, P] = [V, P] \leq C_V(K)$. Hence $[P, K, V] = [K, V] \leq C_V(K) \trianglelefteq V$ by [**5**, Theorem 2.2.3(ii)]. Thus K stabilizes the chain $V \geq C_V(K) \geq 1$ and [**5**, Theorem 5.3.2] forces $K = 1$ to complete the proof of Theorem 3.

It is easy to see that Theorem 3 implies Thompson's $A \times B$ Lemma [**5**, Theorem 5.3.4] when $p = 2$. Suppose in the above that $G = K \times P$, $p = 2$, and $C_V(P) \leq C_V(k)$ for some $k \in K$. Proceed by induction on $|V|$. Since $[V, P] < V$ and G acts on $[V, P]$, we conclude that $[V, P] \leq C_V(k)$. Then Theorem 3 yields the desired conclusion $[V, k] = 1$.

COROLLARY 3.1. *Suppose that $p = 2$ and $[V, P]$ is contained in a characteristic abelian subgroup of V. Let $k \in K$ be such that $\Omega_2(C_V(P)) \leq C_V(k)$. Then $[V, k] = 1$.*

PROOF. Let $1 = M' \leq M$ char V. Thus G acts on M and $\Omega_1(C_M(P)) \leq \Omega_2(C_V(P)) \leq C_V(k)$. Then Theorem 1 implies that $M \leq C_V(k)$ and Theorem 3 yields the desired conclusion.

The next result generalizes and presents an alternate proof of [**3**, Theorem].

THEOREM 4. *Let G be a finite p-group, let E be a subgroup of G and let $\alpha \in \mathrm{Aut}(G)$ be such that $E \leq C_G(\alpha)$. Suppose that $\Omega_1(C_G(E)) \leq C_G(\alpha)$ if $p \neq 2$ and $\Omega_2(C_G(E)) \leq C_G(\alpha)$ if $p = 2$. Then the order of α is a power of p.*

PROOF. Suppose that α is also a p'-element of $\mathrm{Aut}(G)$. Since $\langle \alpha \rangle \times E$ acts on G and $\langle \alpha \rangle$ acts faithfully on $C_G(E)$ by [**5**, Theorem 5.3.4], we conclude that $\alpha = 1$ by Theorem 2 and Corollary 3.1 to complete the proof.

For our final result, we apply Theorems 2 and 3 to obtain generalizations of the fundamental results [**2**, Satz] and [**4**, Theorem 3.1] of H. Bender.

THEOREM 5. *Let H be a p-constrained group, let $Q \in \mathrm{Syl}_p(H)$, let $R = Q \cap O_{p',p}(H)$ and let A be a subgroup of Q. Also let K be an A-invariant p'-subgroup of G and observe that this implies that $[A \cap R, K] \leqslant O_{p'}(H)$. In addition, assume the following two conditions*:

(a) *if $p \neq 2$, then $[\Omega_1(C_R(A)), K] \leqslant O_{p'}(H)$; and*

(b) *if $p = 2$, then $\langle[\Omega_2(C_R(A)), K], [R, A, K]\rangle \leqslant O_{p'}(H)$.*

Then $K \leqslant O_{p'}(H)$.

PROOF. Clearly we may assume that $O_{p'}(H) = 1$ and $R = O_p(H)$. Then KA acts on R, $C_H(R) = Z(R)$ and Theorems 2 and 3 immediately yield the desired conclusion.

REFERENCES

1. R. Baer, *Groups with abelian central quotient group*, Trans. Amer. Math. Soc. **44** (1938), 357–386.

2. H. Bender, *Über den grössten p'-Normalteiler in p-auflösbaren Gruppen*, Arch. Math. (Basel) **18** (1967), 15–16.

3. N. Blackburn, *Automorphisms of finite p-groups*, J. Algebra 3 (1966), 28–29.

4. T. M. Gagen, *Topics in finite groups*, London Math. Soc. Lecture Notes Series, no. 16, Cambridge Univ. Press, London, 1976.

5. D. Gorenstein, *Finite groups*, Harper and Row, New York, 1968.

6. D. Gorenstein and K. Harada, *Finite groups whose 2-subgroups are generated by at most 4 elements*, Mem. Amer. Math. Soc. No. 147 (1974).

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