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S-GROUPS REVISITED

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ABSTRACT. We provide a new characterization of S-groups which is used to develop central results of the theory and, in particular, to show that summands of S-groups are S-groups.

1. Introduction. The theory of S-groups was first developed by Warfield in **[11]**. In that paper, S-groups were introduced as torsion subgroups of balanced projectives (KT-modules), and then characterized as direct sums of certain dense isotype subgroups of totally projective p-groups. The class of S-groups properly contains the class of totally projective p-groups and Warfield was able to extend the classification theory of totally projective groups to this new class. The most important question left open by Warfield was whether summands of S-groups are themselves S-groups. Recently, Stanton **[8]** has answered this in the affirmative.

There is now a satisfactory theory of a class of groups (called Warfield groups) which is a natural extension of the theory of balanced projectives (see, for example, **[3]**). In view of Warfield's definition of S-groups, it is natural to ask whether the torsion subgroup of a Warfield group is an S-group. This was answered in the affirmative in **[2]**. Thus the main results of the theory are the classification theorems (analogues of Ulm's and Zippin's theorems), closure under summands, and the characterization of S-groups as the torsion subgroups of Warfield groups. The latter result could be considered the primary motivation for the study of S-groups.

In this paper we provide (in Theorem 5) a new characterization of S-groups which does not involve balanced projectives, nor the choice of any specific containing groups. This characterization is then used to provide short proofs of the central results described above.

All groups considered will be abelian and local, that is, modules over Z_p, the integers localized at a prime p. Let A be a group. We denote the torsion subgroup of A by A_t and $\mathrm{Ext}(Z(p^\infty), A)$ by $c(A)$. The fundamental facts about $c(A)$ are conveniently summarized in §2 of **[11]**. The group of rational numbers will be denoted by Q. By the rank of A, we mean the rank of $Q \otimes A$ as a Q-vector space. The Ulm invariant of A at the ordinal α will be denoted by $f_A(\alpha)$ and the Ulm invariant of A relative to a subgroup B at α by $f_{A,B}(\alpha)$. We will assume familiarity with the notions and notations of valuated groups developed in **[7]**, and with the local theory of Warfield groups **[3]**.

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2. Tools. Let B be a subgroup of A. Define

$$[B: A] = \{a \in A : p^n a \in B \text{ for some } n < \omega\}.$$

Thus $[B: A]$ is the full inverse image of $(A/B)_t$ under the natural map $A \to A/B$. Let B and C be subgroups of A with $C \subset B$. The following facts will be needed in the sequel.

(A) If A/B is torsion free then $[C: B] = [C: A]$.

(B) If B/C is torsion free divisible then $[B: A]/B \simeq [C: A]/C$.

(C) If B/C is torsion free divisible and C is nice in $[C: A]$ then B is nice in $[B: A]$.

LEMMA 1. *Let A be a group and let α be an ordinal. Then the sequence $0 \to c(p^\alpha A) \to p^\alpha c(A) \to p^\alpha c(A/p^\alpha A) \to 0$ is split exact.*

PROOF. Since p^α is a radical, [6, Lemma 1.1] shows the sequence is exact. The sequence splits because $c(p^\alpha A)$ is cotorsion and $p^\alpha c(A/p^\alpha A)$ is torsion free. □

For each ordinal α, Lemma 1 shows there is a decomposition $p^\alpha c(A) = c(p^\alpha A) \oplus X_\alpha$, where $X_\alpha \simeq p^\alpha c(A/p^\alpha A)$ is cotorsion and torsion free. Set $X = \bigoplus X_\lambda$, where the sum is taken over those limit ordinals not cofinal with ω. The sum $\bigoplus X_\lambda$ is direct because nonzero elements in distinct X_λ's have distinct heights in $c(A)$. We call X a lncwo subgroup of $c(A)$.

If a basic subgroup of each X_λ is chosen, then the resulting basic subgroup B of $\bigoplus X_\lambda$, considered as a valuated subgroup of $c(A)$, is free. By [3, Theorem 31], B is generated by a nice decomposition basis of $[B: c(A)]$ and hence by §2(C), we have X nice in $[X: c(A)]$. For each limit λ not cofinal with ω the rank $k(\lambda, A)$ of the subgroup of B which was chosen in X_λ is clearly an invariant of A which determines B, and hence $c(B)$, as a valuated group. Lemma 1 shows that $k(\lambda, A)$ is the rank of the Z/pZ-vector space $p^\lambda c(A/p^\lambda A)/p^{\lambda+1}c(A/p^\lambda A)$ which is the invariant $k(\lambda, A)$ defined by Warfield [11, p. 158].

3. A characterization of S-groups. Warfield [11] defined an S-group to be the torsion subgroup of a balanced projective group (balanced projectives are also known as KT-modules). We refer the reader to [11] and [3] for a discussion of the properties of balanced projectives. For our purposes, balanced projectives will be characterized as those groups which contain a nice free valuated subgroup with totally projective cokernel. From Warfield's point of view, an S-group comes with a predetermined balanced projective. In this section we characterize S-groups in a way which avoids the choice of specific balanced projectives. The following lemma is the basis for the characterization.

LEMMA 2. *Let A be a reduced torsion group. If for some lncwo subgroup X of $c(A)$ the quotient $[X: c(A)]/X$ is totally projective then A is an S-group.*

PROOF. For each limit ordinal λ not cofinal with ω, choose a p-basic subgroup Y_λ of X_λ and set $Y = \bigoplus Y_\lambda$. Then $[X: c(A)]/X \simeq [Y: c(A)]/Y$ by §2(B), and Y is a free valuated subgroup of $[Y: c(A)]$. By Theorem 31 in [3], Y is nice in $[Y: c(A)]$, so $[Y: c(A)]$ is balanced projective. Since the torsion subgroup of $[Y: c(A)]$ is A, it follows that A is an S-group. □

The next step is to show that all S-groups can be found via Lemma 2. If A is balanced projective then $k(\lambda, A) = 0$ for all limit ordinals λ not cofinal with ω [11]. In particular, when A is totally projective, the only lncwo subgroup of $c(A)$ is trivial. A balanced projective group A is λ-*elementary* if λ is a limit ordinal, A has rank one and $p^\lambda A$ is infinite cyclic. If A is a λ-elementary balanced projective, then $A/p^\lambda A$ is totally projective. Every S-group can be realized as the torsion subgroup of a direct sum of a totally projective group and λ-elementary balanced projectives for various limit ordinals λ not cofinal with ω (for details, see [11]).

LEMMA 3. *Let A be an S-group. Then there is a lncwo subgroup X of $c(A)$ such that $[X: c(A)]/X$ is totally projective.*

PROOF. Let A be the torsion subgroup of $T \oplus N$ where T is totally projective and $N = \bigoplus N_\lambda$, with each N_λ a direct sum of λ-elementary balanced projectives, λ ranging over limit ordinals not cofinal with ω. Since $c(T)$ has only trivial lncwo subgroups, we may assume $T = 0$. Let S_λ be the torsion subgroup of N_λ and set $B = \bigoplus_{\delta \neq \lambda} S_\delta$. Then $p^\lambda c(S_\lambda) = p^\lambda c(N_\lambda) = c(p^\lambda N_\lambda)$. Since $B/p^\lambda B$ is a direct sum of groups of length less than λ, it follows from [6, Lemma 3.10] that $p^\lambda c(B/p^\lambda B) = 0$ and so $p^\lambda c(B) = c(p^\lambda B)$. Thus $p^\lambda c(A) = p^\lambda c(B) \oplus p^\lambda c(S_\lambda) = c(p^\lambda B) \oplus c(p^\lambda N_\lambda) = c(p^\lambda A) \oplus c(p^\lambda N_\lambda)$. Hence $X = \bigoplus c(p^\lambda N_\lambda)$, where the sum is over λ, is a lncwo subgroup of $c(A)$. Noting that $c(A) = c(N)$, and applying §2(A) and §2(B), we have

$$[X: c(A)]/X \simeq \bigoplus \big(c(N_\lambda)/p^\lambda N_\lambda\big)_t = \bigoplus \big(N_\lambda/p^\lambda N_\lambda\big).$$

The latter group is totally projective by definition. □

We have now characterized S-groups as those p-groups A for which $c(A)$ contains a lncwo subgroup X with $[X: c(A)]/X$ totally projective. The characterization is obviously not canonical. However we can show that if A is an S-group, all lncwo subgroups of $c(A)$ are equivalent for our purposes.

LEMMA 4. *Let A be an S-group and let X and Y be lncwo subgroups of $c(A)$. Then there is an automorphism of $c(A)$ which takes X onto Y. In particular, $[X: c(A)]/X \simeq [Y: c(A)]/Y$.*

PROOF. By Lemma 3 we can choose Y so that $[Y: c(A)]/Y$ is totally projective. Let $\{y_i\}$ be a basis for a p-basic subgroup Y_λ^* of Y_λ. Then in the decomposition $p^\lambda c(A) = c(p^\lambda A) \oplus X_\lambda = c(p^\lambda A) \oplus Y_\lambda$ we have $y_i = s_i + x_i$ with $s_i \in c(p^\lambda A)$ and $x_i \in X_\lambda$, and $\{x_i\}$ is a basis for a subgroup X_λ^* of X_λ. Since $c(p^\lambda A)/p^\lambda A$ is torsion free divisible, for each i there is $t_i \in p^\lambda A$ such that $s_i' = s_i - t_i \in p^{\lambda+1}c(A)$. The group C_λ generated by $\{c_i = x_i + t_i\}$ is a free valuated su' group of $c(A)$ isomorphic to X_λ^*. Let $C = \bigoplus C_\lambda$ and $Y^* = \bigoplus Y_\lambda^*$. There is a valuated map $pY^* \to D = \langle\{s_i'\}\rangle$ defined by $py_i \to s_i'$ (this is because we have arranged that $vs_i' > vy_i$). Since $G = [Y^*: c(A)]$ is balanced projective, pG is balanced projective. As pY^* is nice in pG and pG/pY^* is totally projective, we may lift the map $pY^* \to D$ to a map ε: $pG \to c(A)$. Let $\phi = \varepsilon p$: $G \to c(A)$. The cotorsion hull of G is $c(A)$ and G/A is torsion free divisible, so ϕ can be considered an endomorphism of $c(A)$. Let $\psi = 1_{c(A)} - \phi$. Then ψ is an automorphism of $c(A)$ which carries Y^* onto C. It follows that $[C: c(A)]/C \simeq [Y^*: c(A)]/Y^*$. By §2(B) we have $[Y^*: c(A)]/Y^* \simeq [Y: c(A)]/Y$, so $H = [C: c(A)]$ is balanced projective. Let $X^* = \bigoplus X_\lambda^*$. The proof is complete on showing that the isomorphism $C \to X^*$ lifts to an automorphism of $c(A)$. Now $H = [C: c(A)] = [X^*: c(A)]$ by the definition of C, both C and X^* are nice in H, and $f_H = f_{H,C} = f_{H,X^*}$. By [3, Lemma 39], H/X^* is totally projective. Now [9, Theorem 2.8] shows that the valuated isomorphism $C \to X^*$ lifts to an automorphism of H, and hence to an automorphism of $c(A)$. □

Summing up the results of Lemmas 2, 3 and 4, we have:

THEOREM 5. *Let A be a reduced torsion group. Then A is an S-group if and only if $[X: c(A)]/X$ is totally projective for each choice of lncwo subgroup X of $c(A)$. In this case, $[X: c(A)]/X$ is the (unique up to isomorphism) totally projective group with the same Ulm invariants as A.* □

4. Properties of S-groups. The isomorphism theorem is now easy to obtain.

THEOREM 6. *Two S-groups A and B are isomorphic if and only if for each ordinal α we have $f_A(\alpha) = f_B(\alpha)$ and, for each limit ordinal λ not cofinal with ω, we have $k(\lambda, A) = k(\lambda, B)$.*

PROOF. Let X and Y be lncwo subgroups of $c(A)$ and $c(B)$, respectively. Then the equality of k-invariants implies that X and Y are isomorphic as valuated groups. Now X is nice in $[X: c(A)]$ while the Ulm invariants of $[X: c(A)]$ relative to X are just the Ulm invariants of $[X: c(A)]$ and hence of A. The same remarks apply to Y and $c(B)$. By [9, Theorem 2.8], $[X: c(A)]$ and $[Y: c(B)]$ are isomorphic, from which it follows that A and B are isomorphic. □

THEOREM 7. *Each summand of an S-group is an S-group.*

PROOF. Let $A = B \oplus C$ be an S-group. Let X and Y be lncwo subgroups of $c(B)$ and $c(C)$, respectively. Since $c(A) = c(B) \oplus c(C)$, it follows that $X \oplus Y = Z$ is a lncwo subgroup of $c(A)$. Now $[Z: c(A)] = [X: c(B)] \oplus [Y: c(C)]$ and by Theorem 5, $[Z: c(A)]/Z$ is totally projective, so $[X: c(B)]/X$ and $[Y: c(C)]/Y$ are totally projective. Lemma 2 shows that B and C are S-groups. □

5. Torsion subgroups of Warfield groups. We now show that the torsion subgroup of an arbitrary Warfield group is an S-group. Theorem 7 provides a straightforward reduction to the rank one case. From here the plan is to construct a Warfield group of rank one with an appropriate torsion subgroup and then apply the isomorphism theorem for Warfield groups. First some definitions.

The *value sequence* $V_G a$ (or just Va if the context is clear) *of an element a* of a valuated group G is the sequence $va, vpa, vp^2a, \cdots$. A *value sequence* α is a sequence $\alpha_0 < \alpha_1 < \alpha_2 < \ldots$ of ordinals and symbols ∞. By $p^n\alpha$ we mean the value sequence $\alpha_n, \alpha_{n+1}, \ldots$. Value sequences α and β are *equivalent* if there are natural numbers m and n such that $p^m\alpha = p^n\beta$. If α is an ordinal or symbol ∞, we identify α with the value sequence $\alpha, \alpha + 1, \alpha + 2, \ldots$. Value sequences of finite order elements will be considered finite in the obvious way.

Although [2, Theorem 5.21] states that the torsion subgroup of a rank one Warfield group is an S-group, the proof relies on [1, Theorem 103.3] which is incorrect. To see this, let T be a totally projective group of length $\omega_1 + \omega$. Then T and the value sequence ω_1 satisfy conditions (i)–(iv) of [1, pp. 200–201], so [1, Theorem 103.3] claims the existence of a rank one group G with $G_t = T$ such that G contains an element with value sequence ω_1. But G is Warfield and the torsion subgroup of a rank one Warfield group which contains an element with value sequence ω_1 is not totally projective, a contradiction to the choice of T. Theorem 8 is our replacement for [1, Theorem 103.3]. The error which leads to the preceding counterexample lies in part (c), [1, p. 202]. The proof of part (d) on the same page also seems to be incorrect. For example, if the value sequence in question is 0, 2, 4, 6, . . . then the sequence of x_i's which Fuchs requires cannot be obtained. Koyama [4] also provides corrections to [1, Theorem 103.3]. The following provides a considerable simplification of both the theorem statement and proof.

THEOREM 8. *Let $C = \langle x \rangle$ be a valuated cyclic group of infinite order and let T be a reduced torsion group. Then there is a rank one group G with $G_t = T$ and containing C as a valuated subgroup if and only if*:

(a) $f_T \geqslant f_C$;

(b) *if Vx is equivalent to an ordinal $\beta \geqslant \omega$ then*

$$p^\beta c(T/p^\beta T) \neq 0.$$

PROOF. Let $Vx = \alpha = \alpha_0, \alpha_1, \ldots$. For necessity, the only difficulty is when α is equivalent to an ordinal $\geqslant \omega$. In this case we may assume that α is an ordinal. Under the natural embedding of G as a valuated subgroup of $c(T)$ we have $x \in p^\alpha c(T)$. Write $p^\alpha c(T) = c(p^\alpha T) \oplus X$. Since $c(p^\alpha T)/p^\alpha T$ is divisible, no element of infinite order in $c(p^\alpha T)$ has value sequence equivalent to α. Thus $X \simeq p^\alpha c(T/p^\alpha T)$ is not zero.

Now for the proof of sufficiency. Let $I = \{i: \alpha_i + 1 < \alpha_{i+1}\}$. If α contains ∞, it suffices to take $G = T \oplus Q$. When α does not contain ∞, we distinguish two cases.

Case 1: *I is finite*. We may as well assume that I is empty. Suppose Vx is equivalent to the ordinal β. If $\beta < \omega$, we may assume $\beta = 0$ and set $G = T \oplus Z_p$. If $\beta \geqslant \omega$, condition (b) ensures that $p^\beta c(T) = c(p^\beta T) \oplus X_\beta$ with $X_\beta \simeq p^\beta c(T/p^\beta T) \neq 0$. Let y be an element of X_β with value β and let $G = [\langle y \rangle: c(T)]$. Since $c(T)/G$ is torsion free, G is isotype in $c(T)$ and so $V_G y = \beta$. Clearly $G_t = T$ and G contains an element with value sequence α.

Case 2: *I is infinite*. Let $\lambda = \sup \alpha_i$. Condition (a) ensures there is a sequence of x_i's in T so that $Vx_i = \alpha_0, \ldots, \alpha_i$ and it is straightforward to further arrange that $\lim_{i\to\infty} v(x_{i+1} - x_i) = \lambda$. As λ is cofinal with ω, [5, Theorem 3.4] shows that $c(T)/p^\lambda c(T)$ is complete in the p^λ-topology. Then $x = \lim x_i$ has value sequence α in $c(T)/p^\lambda c(T)$ and so lifts to an element y of value sequence α in $c(T)$. Setting $G = [\langle y \rangle: c(T)]$ completes the proof. □

THEOREM 9. *The torsion subgroup of a Warfield group is an S-group.*

PROOF. Let W be the Warfield group in question. By [3, Lemma 15], there is a totally projective p-group P such that $W \oplus P$ is a direct sum of groups with torsion free rank one. By Theorem 7, we may assume that W has torsion free rank one. If W is balanced projective, there is nothing to prove. If not, the number of jumps in the value sequence of an element a of infinite order in W is infinite. Let T be the totally projective p-group with the same Ulm invariants as W. By Theorem 8 there is a rank one Warfield group A with torsion part T and containing an element with value sequence the same as the value sequence of a. The isomorphism theorem for Warfield groups [3, Theorem 10] shows that A and W are isomorphic. Thus W_t is totally projective. □

REFERENCES

1. L. Fuchs, *Infinite abelian groups*, vol. II, Academic Press, New York, 1973.

2. R. H. Hunter, *Balanced subgroups of abelian groups*, Trans. Amer. Math. Soc. **215** (1976), 81–98.

3. R. Hunter, F. Richman and E. Walker, *Warfield modules*, Abelian Group Theory, Proc. 2nd New Mexico State Univ. Conf. (1976), Lecture Notes in Math., vol. 616, Springer-Verlag, Berlin and New York, 1977, pp. 87–123.

4. T. Koyama, *On p-indicators in* Ext(Q/Z, T), Hiroshima Math. J. **8** (1978), 211–216.

5. R. Mines, *A family of functors defined on generalized primary groups*, Pacific J. Math. **26** (1968), 349–360.

6. R. J. Nunke, *Homology and direct sums of countable abelian groups*, Math. Z. **101** (1967), 182–212.

7. F. Richman and E. Walker, *Valuated groups*, J. Algebra **56** (1979), 145–167.

8. R. O. Stanton, *S-groups* (preprint).

9. E. A. Walker, *Ulm's theorem for totally projective p-groups*, Proc. Amer. Math. Soc. **37** (1973), 387–392.

10. K. D. Wallace, *On mixed groups of torsion-free rank one with totally projective primary components*, J. Algebra **17** (1971), 482–488.

11. R. B. Warfield, Jr., *A classification theorem for abelian p-groups*, Trans. Amer. Math. Soc. **210** (1975), 149–168.

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