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# A RING WITH ARITHMETICAL CONGRUENCE LATTICE NOT PRESERVED BY ANY PIXLEY FUNCTION

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ABSTRACT. A ring $(A;\ +,\ \cdot)$ is constructed such that the congruence lattice $L_A$ of the ring $(A;\ +,\ \cdot)$ is distributive, the elements of $L_A$ are pairwise permutable and there is no $L_A$-compatible function $p$ on $A$ such that

$$p(a, b, b) = p(a, b, a) = p(b, b, a) = a \quad \text{for all } a, b \in A. \tag{1}$$

**1. Introduction and notation.** Let $\mathrm{Eq}(A)$ denote the lattice of all equivalence relations on a set $A$. A sublattice $L$ of $\mathrm{Eq}(A)$ is said to be arithmetical if $L$ is distributive and the elements of $L$ are pairwise permutable. For arbitrary $\mathfrak{D} \in \mathrm{Eq}(A)$ and $(a_1, \ldots, a_k), (b_1, \ldots, b_k) \in A^k$ we write

$$(a_1, \ldots, a_k)\mathfrak{D}(b_1, \ldots, b_k) \tag{1.1}$$

if $a_1\mathfrak{D}b_1 \wedge \cdots \wedge a_k\mathfrak{D}b_k$. A mapping $f$ of [a subset of] $A^k$ into $A$ is said to be a $k$-ary $\mathfrak{D}$-compatible [partial] function if (1.1) implies

$$f(a_1, \ldots, a_k)\mathfrak{D}f(b_1, \ldots, b_k) \tag{1.2}$$

for all $(a_1, \ldots, a_k)$, $(b_1, \ldots, b_k) \in \mathrm{Dom}(f)$. For $L \subseteq \mathrm{Eq}(A)$, $f$ is said to be $L$-compatible if it is $\mathfrak{D}$-compatible for every $\mathfrak{D} \in L$. A ternary function $p$ on the set $A$ is said to be a Pixley function if

$$p(a, b, b) = p(a, b, a) = p(b, b, a) = a \quad \text{for all } a, b \in A. \tag{1.3}$$

Let an arithmetical lattice $L \subseteq \mathrm{Eq}(A)$ be the congruence lattice of an algebra over $A$. It was proved in [9] that if $L$ is finite then there is an $L$-compatible Pixley function. At the Colloquium on Universal Algebra in Oberwolfach, July, 1973, A. F. Pixley asked whether the finiteness of $L$ can be omitted in the mentioned theorem. Partial positive answers are contained in [7] and [8]; e.g. the finiteness of $L$ can be replaced by the countability of $A$. In the present paper it is shown that the finiteness of $L$ cannot be completely omitted.

A related problem is solved in [3] where congruence equalities are defined; for example, arithmeticity corresponds to the congruence equality $(\xi\,\eta) \cap (\zeta\,\xi) = \xi(\eta \cap \zeta)$. Further, local Mal'cev characterisability is introduced, and it is proved that arithmeticity is the only nontrivial congruence equality which is locally Mal'cev characterisable.

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$L_Z$ denotes the congruence lattice of the ring $(Z, +, \cdot)$ of integers and $\omega$ the set of nonnegative integers. The symbol

$$\binom{n}{i} = \frac{n(n-1)\cdots(n-i+1)}{i!} \tag{1.4}$$

is defined for all $n \in Z$ and $i \in \omega$. The (nonnegative) least common multiple of integers $a_1, \ldots, a_n$ is denoted l.c.m.$(a_1, \ldots, a_n)$. Square brackets are used for the integer part. Sequences of integers (i.e. mappings of $\omega$ into $Z$) are denoted $x, y, z, u, v, t \ldots$. The $n$th member of the sequence $x$ is denoted $x_n$; every sequence begins with the 0th member. Hence $x = (x_0, x_1, x_2, \ldots)$, and analogously for other letters. The formula

$$x \equiv y \pmod{t} \tag{1.5}$$

means $x_n \equiv y_n \pmod{|t_n|}$ for all $n \in \omega$; $a \equiv b \pmod 0$ means $a = b$; and $a \equiv b \pmod 1$ holds for arbitrary integers $a, b$. We also generalize (1.5) for $k$-tuples of sequences similarly to (1.1).

**2. Lower bound for $L_Z$-compatible Pixley functions.** We shall need some results from [4], which are summarized in the next theorem.

2.1. THEOREM. *Let $f$ be a mapping of $\omega$ into $Z$. Then $f$ can be uniquely represented in the form*

$$f(k) = \sum_{i=0}^{\infty} A_i \binom{k}{i}. \tag{2.1.1}$$

*The numbers $A_i$ are integers and for every $n \in \omega$*

$$A_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(n-i). \tag{2.1.2}$$

*Further, $f$ is $L_Z$-compatible if and only if for every $n \in \omega$ the number $A_n$ is a multiple of* l.c.m.$(1, 2, \ldots, n)$.

The formula (2.1.2) can be proved from (2.1.1) and also by [6, formula 34, p. 438].

The paper [4] also contains a lower bound for so-called genuine pseudopolynomials. There are several reasons why the bound from [4] cannot be immediately used here; the most substantial one is that the bound in [4] is not uniform. We shall find a uniform exponential lower bound for $L_Z$-compatible Pixley functions; however, we shall not look for the best numerical result. To formulate our result briefly we define

2.2. DEFINITION. Let $M_0 = 0$, $M_1 = 1$ and for all $n \in \omega$, $n > 2$ let

$$M_n = \left[\text{l.c.m.}(1, 2, \ldots, k)/2^k\right] \tag{2.2.1}$$

where $k$ is the greatest prime not greater than $n$.

By [4] or [1] the sequence $(M_0, M_1, M_2, \ldots)$ increases exponentially. More precisely,

$$\lim_{n\to\infty} \log(2^n \cdot M_n)/\log(e^n) = 1 \tag{2.2.2}$$

where $e = 2.71828\ldots$.

2.3. THEOREM. For every $L_Z$-compatible Pixley function $p$ and every $n \in \omega$

$$\max\{|p(a, b, c)|;\ a, b, c \in \{-n, -n+1, \dots, n-1, n\}\} \geqslant M_n. \quad (2.3.1)$$

PROOF. For $n \leqslant 1$ we have $|p(n, n, n)| = |n| = n = M_n$, hence (2.3.1) holds. Now assume $n > 1$; without loss of generality we may assume that $n$ is a prime. Denote

$$f(k) = p(k, k-n, 0). \quad (2.3.2)$$

It obviously suffices to show

$$\max\{|f(k)|;\ k \in \{0, 1, \dots, n\}\} \geqslant M_n. \quad (2.3.3)$$

The partial function $f$ is $L_Z$-compatible and hence for every $k \in \omega$ we have $f(k) = p(k, k-n, 0) \equiv p(k, k, 0) = 0 \pmod{n}$. Further, $f(0) = p(0, -n, 0) = 0$, $f(n) = p(n, 0, 0) = n$. There are integers $A_i$ such that (2.1.1) holds for all $k \in \omega$; then (2.1.2) holds, too. For every $i \in \{1, 2, \dots, n-1\}$ the prime $n$ divides both $\binom{n}{i}, f(n-i)$ and hence

$$\binom{n}{i} f(n-i) \equiv 0 \pmod{n^2}.$$

Therefore

$$A_n \equiv f(n) + (-1)^n f(0) = n \pmod{n^2}.$$

Hence $A_n \neq 0$. On the other hand, (2.1.2) implies

$$|A_n| \leqslant \sum_{i=0}^{n} \binom{n}{i} f(n-i) < 2^n \max\{|f(k)|;\ k \in \{0, 1, \dots, n\}\}.$$

If (2.3.3) does not hold then we have

$$0 \neq |A_n| < |2^n \cdot M_n| \leqslant \text{l.c.m.}\,(1, 2, \dots, n).$$

Therefore $A_n$ is not a multiple of l.c.m.$(1, 2, \dots, k)$ which contradicts the $L_Z$-compatibility of $f$.

**3. The example.** A sequence of integers $(x_0, x_1, x_2, \dots)$ is said to be polynomially bounded if there is $k \in \omega$ such that $|x_n| < (n+2)^k$ for all $n \in \omega$. The memberwise product and the memberwise difference of two polynomially bounded sequences of integers are polynomially bounded. Therefore the set of polynomially bounded sequences forms a ring with these operations. This ring will be called the ring of polynomially bounded sequences.

3.1. THEOREM. *Let $L_A$ be the congruence lattice of the ring $(A;\ +,\ \cdot)$ of polynomially bounded sequences. Then $L_A$ is arithmetical and there is no $L_A$-compatible Pixley function.*

The proof will be divided into several lemmas. Throughout the whole present section $A$ and $L_A$ will always have the meaning from Theorem 3.1.

3.2. LEMMA. *$L_A$ is arithmetical.*

PROOF. The elements of $L_A$ are obviously pairwise permutable; it remains to prove that $L_A$ is distributive.

Let $\xi, \eta, \zeta \in L_A$, $x, y, z \in A$ and $x\xi y$, $x\eta z\zeta y$. We have to find $u \in A$ such that

$$x(\xi \cap \eta)u(\xi \cap \zeta)y. \tag{3.2.1}$$

For every $n \in \omega$ consider the system of congruences

$$\begin{aligned} u_n &\equiv x_n \quad (\operatorname{mod}|x_n - y_n|), \\ u_n &\equiv x_n \quad (\operatorname{mod}|x_n - z_n|), \\ u_n &\equiv y_n \quad (\operatorname{mod}|y_n - z_n|), \end{aligned} \tag{3.2.2}$$

with the unknown $u_n$. Every pair of congruences (3.2.2) is solvable. Hence by the Chinese remainder theorem the system (3.2.2) is solvable. Moreover, if the integers $x_n, y_n, z_n$ are pairwise different we can arrange

$$|u_n| < |(x_n - y_n) \cdot (x_n - z_n) \cdot (y_n - z_n)|;$$

otherwise $u_n$ will be the repeated integer from $x_n, y_n, z_n$. Then the sequence $u = (u_0, u_1\, u_2, \ldots)$ is polynomially bounded and hence $u \in A$. Since (3.2.2) holds for all $n \in \omega$ we have $u\xi x$, $u\eta x$, $u\zeta y$ which together with $x\xi y$ implies (3.2.1). Q.E.D.

The next lemma obviously holds for any $k$-ary $L_A$-compatible function; since we need it only for $k = 3$ we formulate it only for this case to avoid more complicated notation.

3.3. Lemma. *For every ternary $L_A$-compatible function $p$ there is a unique sequence $(p_0, p_1, p_2, \ldots)$ of ternary $L_Z$-compatible functions such that*

$$\begin{aligned} &p((x_0, x_1, x_2, \ldots), (y_0, y_1, y_2, \ldots), (z_0, z_1, z_2, \ldots)) \\ &\qquad = (p_0(x_0, y_0, z_0), p_1(x_1, y_1, z_1), p_2(x_2, y_2, z_2), \ldots) \end{aligned} \tag{3.3.1}$$

for all $(x_0, x_1, x_2, \ldots), (y_0, y_1, y_2, \ldots), (z_0, z_1, z_2, \ldots) \in A$.

Proof. For every $n \in \omega$ denote by $\mathfrak{D}_n$ the least element of $L_A$ such that

$$(1, \ldots, 1, 0, 1, 1, \ldots)\mathfrak{D}_n(0, 0, 0, \ldots) \tag{3.3.2}$$

where 0 on the left-hand side is at the $n$th place. Since the function $p$ is $\mathfrak{D}_n$-compatible the $n$th member of $p(x, y, z)$ depends only on the $n$th members $x_n, y_n, z_n$ of $x, y, z$, respectively. Therefore there are ternary functions $p_0, p_1, p_2, \ldots$ such that (3.3.1) holds; the uniqueness is obvious. It remains to prove that the functions $p_0, p_1, p_2, \ldots$ are $L_Z$-compatible. Let $a_1, a_2, a_3, b_1, b_2, b_3 \in Z$, $n, d \in \omega$ and $(a_1, a_2, a_3) \equiv (b_1, b_2, b_3) \pmod d$. Let the $n$th members of the sequences $x, y, z, u, v, w, t$ be $a_1, a_2, a_3, b_1, b_2, b_3, d$, respectively, and let all the remaining members of these sequences be zero. Then

$$(x, y, z) \equiv (u, v, w) \pmod t$$

and since $p$ is $L_A$-compatible

$$p(x, y, z) \equiv p(u, v, w) \pmod t.$$

Therefore by (3.3.1)

$$p_n(x_n, y_n, z_n) \equiv p_n(u_n, v_n, w_n) \pmod{t_n}$$

i.e.

$$p_n(a_1, a_2, a_3) \equiv p_n(b_1, b_2, b_3) \pmod{d}.$$

Q.E.D.

3.4. LEMMA. *There is no $L_A$-compatible Pixley function.*

PROOF. Let $p$ be such a function. By Lemma 3.3 there are $L_Z$-compatible functions $p_0, p_1, p_2, \ldots$ such that (3.3.1) holds; $p_0, p_1, p_2, \ldots$ are obviously Pixley functions. Hence for every $n \in \omega$ there are integers $x_n, y_n, z_n \in \{-n, -n + 1, \ldots, n - 1, n\}$ such that

$$|p_n(x_n, y_n, z_n)| > M_n.$$

The sequences $x = (x_0, x_1, x_2, \ldots)$, $y = (y_0, y_1, y_2, \ldots)$, $z = (z_0, z_1, z_2, \ldots)$ belong to $A$, hence $t = (t_0, t_1, t_2, \ldots) = p(x, y, z)$ also belongs to $A$. However, for every $n \in \omega$, $|t_n| = |p_n(x_n, y_n, z_n)| > M_n$. Hence $t$ is not polynomially bounded, which contradicts $t \in A$.

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