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DECOMPOSING OVERRINGS

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ABSTRACT. We show that if $S \supset R$ are rings such that S_R is projective, then R_R is a direct summand of S_R if and only if S_R is faithfully projective (this condition holds, in particular, if S_R is free).

1. Introduction. A common problem in homological algebra is to relate the global dimension of a subring R of a ring S to the global dimension of S. Typically, S will be projective as an R-module, but frequently it is also required that R be a direct summand of S (see for example [2], [3]). This note shows that the second requirement is ensured by a fairly weak condition, which is also trivially necessary. In particular, the condition holds whenever S is free as an R-module. In the special case when R is commutative and S_R is finitely generated, the result has been noticed by Bourbaki [1, Ex. 5.4, p. 176].

All modules are right modules, unless otherwise stated. All rings will contain an identity, and this identity will be preserved when passing to subrings. Let R be a ring, and P a right R-module. P is said to be *faithfully projective* over R, if P is projective and $PM \neq P$ for every maximal two-sided ideal M of R.

2. THEOREM 1. *Let R be a subring of a ring S such that S_R is projective. Then $S_R \cong R \oplus S'$ for some module S' if and only if S_R is faithfully projective.*

PROOF. If S_R is projective and R is a direct summand of S, then it is a triviality to prove that S_R is faithfully projective.

Conversely, assume that S_R is faithfully projective. Let G be the direct sum of a sufficiently large, infinite number of copies of R so that G_R maps onto S_R. Then ${}_RF_R = {}_RS_R \oplus {}_RG_R$ is an R-bimodule which, as a right R-module, is free with basis, say, $\{e_i\}$. Let $1 \in R \subset S$ have representation $1 = \sum_1^m e_ir_i$ in this basis. We shall consider the left ideal

$$O(1) = \{\phi(1)\colon \phi \in \operatorname{Hom}(F, R)\}.$$

Since $\operatorname{Hom}(F, R) = \prod R\phi_i$, where $\phi_i(e_j) = \delta_{ij}$, we have $O(1) = \prod R\phi_i(1) = \sum_1^m Rr_i$.

We first want to show that $O(1)$ is a two-sided ideal. Since F is an R-bimodule, $fe_i \in F$ for all $f \in R$. Thus $fe_i = \sum_j e_jf_{ij}$ from some $f_{ij} \in R$ and

$$f \cdot 1 = f \cdot \left(\sum e_ir_i\right) = \sum_{i,j} e_jf_{ij}r_i.$$

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On the other hand, $f \cdot 1 = 1 \cdot f = \sum e_j r_j f$. Identifying coefficients gives

$$r_j f = \sum_i f_{ij} r_i \in \sum Rr_i,$$

and consequently $r_j R \subseteq \sum Rr_i$ for each $1 < j < m$. Thus $O(1)$ is indeed a two-sided ideal.

Suppose that $O(1) \neq R$. Then $O(1)$ is contained in some maximal two-sided ideal, say M. Notice that $SM \cap R \neq R$. (If $SM \cap R = R$, then $S = S\,(SM \cap R) \subseteq SM$, contradicting the faithful projectivity of S.) Thus $SM \cap R = M$. But this implies that $FM \cap R = FM \cap S \cap R = SM \cap R = M$. Thus

$$1 = \sum e_i r_i \in F \cdot O(1) \cap R \subseteq FM \cap R = M;$$

which is clearly absurd. Therefore $O(1) = R$.

Thus there exists a homomorphism $\phi\colon F \to R$, or its restriction $\varphi\colon S \to R$, such that $\varphi(1) = 1$. Consequently R is a direct summand of S.

3. Examples.

(3.1) The following is an example of a projective, but not faithfully projective ring extension. Let

$$R = \begin{pmatrix} \mathbf{Z} & 2\mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} \end{pmatrix} \subseteq S = \begin{pmatrix} \mathbf{Z} & \mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} \end{pmatrix}.$$

Then **R** is a hereditary Noetherian prime ring and is the idealizer of the right ideal

$$I = \begin{pmatrix} 2\mathbf{Z} & 2\mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} \end{pmatrix}$$

of S. Furthermore $SI = S$. Therefore S_R is projective but not faithfully projective, and thus R cannot be a direct summand of S.

(3.2) The following example, suggested by a remark of Murray Schacher, provides a ring extension $R \subseteq S$ such that S_R is free, but S/R is not free. Let $R = \mathbf{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$ and let P be the kernel of the homomorphism:

$$\varphi\colon R^3 \to R; \qquad \varphi(a, b, c) = xa + yb + zc.$$

Since φ is surjective, P is projective, and it is well-known that P is not free (see, for example, [**3**, p. 30]). Let S be the trivial extension of R, that is,

$$S = \left\{ \begin{pmatrix} r & p \\ 0 & r \end{pmatrix} \middle| \, r \in R, p \in P \right\} \supset R = \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \middle| \, r \in R \right\}.$$

Then, as an R-module S splits; $S_R \cong R \oplus P$. So $S_R \cong R^{(3)}$ is free but $S/R \cong P$ is not.

REMARK. The situation illustrated by Example 3.2 cannot occur when S is an infinite-dimensional free R-module, since the "stably free" implies "free" [**3**, Proposition 4.2]. Similarly, if S has a "large" rank as an R-module, then S/R must be free; precisely, if R is Noetherian and the rank of S is larger than the Krull dimension of R (in the sense of Rentschler-Gabriel), then it follows from [**5**] that S/R must be free.

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