

SUBORDINATION BY UNIVALENT FUNCTIONS

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ABSTRACT. Let K be the class of functions $f(z) = z + a_2z^2 + \dots$, which are regular and univalently convex in $|z| < 1$. In this paper we establish certain subordination relations between an arbitrary member f of K , its partial sums and the functions $(\lambda/z) \int_0^z f(t)dt$ and $\mu \int_0^z t^{-1}f(t)dt$. The well-known result that $z/2$ is subordinate to $f(z)$ in $|z| < 1$ for every f belonging to K follows as a particular case from our results. We also improve certain results of Robinson regarding subordination by univalent functions. A sufficient condition for a univalent function to be convex of order α is also given.

Introduction. Let A denote the class of functions $f(z) = z + a_2z^2 + \dots$ which are regular in $|z| < 1$. We denote by S the subclass of A consisting of functions f which are univalent in $|z| < 1$; S^* and K will stand for the usual subclasses of S whose members are, respectively, starlike (w.r.t. the origin) and convex in $|z| < 1$. A function f belonging to A is said to be convex of order α , $0 \leq \alpha < 1$, in $|z| < 1$ if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (|z| < 1), \quad (1)$$

and we shall denote by $K(\alpha)$ the class of functions satisfying (1).

A function f is said to be subordinate to a function F (in symbols $f(z) \prec F(z)$) in $|z| < r$ if there exists a regular function $w(z)$ with $|w(z)| < |z| < r$, such that $f(z) = F(w(z))$ in $|z| < r$. For F univalent in $|z| < r$, $f(z) \prec F(z)$ in $|z| < r$ is equivalent to $f(0) = F(0)$ and $f(|z| < r) \subset F(|z| < r)$.

In the sequel whenever we come across the notation $f(z) \prec F(z)$ for $|z| < r$ we shall understand that the superordinate function F is univalent in $|z| < r$ and $f(0) = F(0)$.

The Hadamard product or convolution of two power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series $\sum_{n=0}^{\infty} a_n b_n z^n$ and denoted by $(f * g)(z)$.

A sequence $\{b_n\}_1^{\infty}$ of complex numbers is called a subordinating factor sequence if, whenever $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is univalent and convex in $|z| < 1$, we have

$$\sum_{n=1}^{\infty} a_n b_n z^n \prec f(z)$$

in $|z| < 1$.

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The de la Vallée Poussin means of a regular function $f(z) = a_1z + a_2z^2 + \dots$ are defined by

$$V_n(z, f) = \left(\frac{2n}{n}\right)^{-1} \sum_{k=1}^n \binom{2n}{n+k} a_k z^k,$$

for $n = 1, 2, 3, \dots$. Pólya and Schoenberg [5, Theorem 2, p. 298] showed that these means are convex (starlike) if and only if f is convex (starlike).

Robinson found a number r_0 , $0 < r_0 < 1$, with the property that if f and F are regular for $|z| < 1$, $f(0) = F(0)$ then the relation $zf'(z) < zF'(z)$ in $|z| < 1$ implies that $f(z) < F(z)$ at least for $|z| < r_0$ [8, Theorem, p. 20]. In our first theorem we improve this result. Theorem 2 improves a similar result of Robinson [8, Theorem, p. 22]. In Theorem 3 we give a sufficient condition, involving the principle of subordination, for a univalent function to be convex of order α . Theorems 4 and 5 deal with subordination relations between an arbitrary element f of K and its "convex transformations" $(\lambda/z) \int_0^z f(t) dt$ and $\mu \int_0^z t^{-1} f(t) dt$, where λ, μ are some positive real numbers. Theorem 6 generalizes the well-known result $z/2 < f(z)$ for every member f of K and the last one deals with a subordination relation between the de la Vallée Poussin mean of second order and the second partial sum of a normalized convex univalent function.

The following result due to Suffridge [10] will be used to prove Theorem 1.

LEMMA 1. *If $f \in K$, $g(z) = \sum_{n=1}^{\infty} b_n z^n$ is regular in $|z| < 1$, and $zg'(z) < zf'(z)$ for $|z| < 1$, then $g(z) < f(z)$.*

THEOREM 1. *If $f(z)$ and $F(z)$ are regular for $|z| < 1$, $zF'(z)$ is univalent in $|z| < 1$, $f(0) = F(0)$ and $zf'(z) < zF'(z)$, then*

$$f(z) < F(z) \quad \text{at least for } |z| < r_0, \quad (2)$$

where $r_0 = \tanh \pi/4 = 0.655 \dots$

PROOF. Since $zF'(z)$ is univalent in $|z| < 1$, it is starlike for $|z| < \tanh \pi/4 = r_0$ (say) and hence $F(z)$ is convex in this disc. From this it follows that $F(r_0z)$ is convex in $|z| < 1$. Thus our hypothesis implies that

$$zf'(r_0z) < zF'(r_0z), \quad |z| < 1,$$

where now $F(r_0z)$ is convex in $|z| < 1$. From this, using Lemma 1, we conclude that

$$f(r_0z) < F(r_0z)$$

in $|z| < 1$ and hence

$$f(z) < F(z)$$

at least for $|z| \leq r_0$, where $r_0 = \tanh \pi/4$. This completes the proof of our theorem.

Robinson [8] was able to establish the relation (2) only for $|z| < 1/5$. We have thus considerably improved his result.

The following result of Ruscheweyh and Sheil-Small [9, Theorem 4.1] finds an application in our next theorem.

LEMMA 2. *Let ϕ and ψ be convex in $|z| < 1$ and suppose that f is subordinate to ψ . Then $(\phi * f)(z) < (\phi * \psi)(z)$, $|z| < 1$.*

THEOREM 2. *If $g(z)$ and $G(z)$ are regular for $|z| < 1$, $[zG(z)]'$ is univalent in $|z| < 1$ and $[zg(z)]'$ is subordinate to $[zG(z)]'$ in $|z| < 1$, then*

$$g(z) < G(z) \quad (3)$$

at least for $|z| < 2 - \sqrt{3} = 0.268 \dots$

PROOF. We define f and F by the relations

$$2f(z) = [zg(z)]' \quad \text{and} \quad 2F(z) = [zG(z)]'.$$

Thus we are given that

$$2f(z) < 2F(z)$$

or

$$f(z) < F(z), \quad |z| < 1.$$

We have to prove that

$$g(z) = \frac{2}{z} \int_0^z f(t) dt < \frac{2}{z} \int_0^z F(t) dt = G(z),$$

at least for $|z| < 2 - \sqrt{3}$.

Since the function $k(z) = z/(1-z)$ belongs to K , the function

$$h(z) = \frac{2}{z} \int_0^z \frac{t}{1-t} dt = \sum_{n=1}^{\infty} \frac{2}{n+1} z^n$$

also belongs to K [4].

The function F being univalent in $|z| < 1$ is convex in the disc $|z| < 2 - \sqrt{3} = r_0$ (say). Thus we are given that $f(z) < F(z)$ in $|z| < 1$ and $F(z)$ is convex in $|z| < r_0$. From this we have that

$$f(r_0 z) < F(r_0 z) \quad (|z| < 1),$$

with $F(r_0 z)$ convex in $|z| < 1$. In view of Lemma 2 it follows that

$$h(z) * f(r_0 z) < h(z) * F(r_0 z)$$

in $|z| < 1$, which is the same as

$$g(r_0 z) < G(r_0 z)$$

in $|z| < 1$ and hence

$$g(z) < G(z)$$

at least for $|z| < r_0 = 2 - \sqrt{3}$.

It was earlier proved by Robinson [8] that the assertion (3) holds at least in the disc $|z| < 1/5$.

In the next theorem we will use the following result of Robertson [6] to determine a sufficient condition for a univalent function to be in $K(\alpha)$.

LEMMA 3. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be regular and univalent in $|z| < 1$. For $0 < t < 1$ let $F(z, t)$ be regular in $|z| < 1$. Let $F(z, 0) \equiv f(z)$ and $F(0, t) \equiv 0$. Let p be a positive real number for which*

$$F(z) = \lim_{t \rightarrow 0+} \left[\frac{F(z, t) - F(z, 0)}{zt^p} \right]$$

exists. Let $F(z, t)$ be subordinate to $f(z)$ in $|z| < 1$ for $0 \leq t \leq 1$. Then

$$\operatorname{Re} \left\{ \frac{F(z)}{f'(0)} \right\} < 0, \quad |z| < 1.$$

If in addition $F(z)$ is also analytic in $|z| < 1$ and $\operatorname{Re} F(0) \neq 0$ then $\operatorname{Re}\{F(z)/f'(z)\} < 0$ for $|z| < 1$.

THEOREM 3. If f belongs to S and

$$F(z, t) = \frac{1}{2} \left(1 - \frac{i\alpha t}{2} \right) f(ze^{it}) + \frac{1}{2} \left(1 + \frac{i\alpha t}{2} \right) f(ze^{-it}) < f(z) \quad (5)$$

for $0 \leq t \leq 1$, $|z| < 1$, and fixed real number α , $\alpha < 1$, then f belongs to $K(\alpha)$.

PROOF. Clearly $F(z, 0) = f(z)$, $F(0, t) = 0$. If

$$F(z) = \lim_{t \rightarrow 0+} \left[\frac{F(z, t) - F(z, 0)}{zt^2} \right],$$

then a simple calculation shows that

$$\begin{aligned} F(z) &= \lim_{t \rightarrow 0+} \left[\frac{\partial F(z, t)/\partial t}{2zt} \right] = \lim_{t \rightarrow 0+} \left[\frac{1}{2z} \frac{\partial^2 F(z, t)}{\partial t^2} \right] \\ &= -\frac{1}{2} [zf''(z) + f'(z) - \alpha f'(z)]. \end{aligned} \quad (6)$$

Since $F(z)$ is regular in $|z| < 1$ and $\operatorname{Re} F(0) = -(1 - \alpha)/2 \neq 0$, by the above lemma we have $\operatorname{Re}[F(z)/f'(z)] < 0$ and consequently

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, \quad |z| < 1,$$

that is, $f \in K(\alpha)$.

REMARK. If $\alpha = 0$ the subordination condition (5) becomes

$$(1/2)[f(ze^{it}) + f(ze^{-it})] < f(z) \quad (7)$$

which is the result obtained by Robertson [6].

The following three theorems depend on a result due to Wilf [11] that characterizes subordinating factor sequences and, in particular, asserts that if

$$\operatorname{Re} \left\{ 1 + 2 \sum_{n=1}^{\infty} b_n z^n \right\} > 0 \quad \text{for } |z| < 1$$

then $\{b_n\}$ is a subordinating factor sequence.

THEOREM 4. If f belongs to K and g is defined by

$$g(z) = \frac{\lambda}{z} \int_0^z f(t) dt, \quad (8)$$

then for every λ , $0 < \lambda < 1/2(1 - \log 2) = 1.629 \dots$, we have

$$g(z) < f(z) \quad (|z| < 1), \quad (9)$$

and this result is sharp.

PROOF. Suppose $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then

$$g(z) = \frac{\lambda}{2} z + \sum_{n=2}^{\infty} \frac{\lambda}{n+1} a_n z^n. \quad (10)$$

In view of Wilf's result, the assertion (9) will hold for $|z| < 1$, if

$$\operatorname{Re} \left\{ 1 + 2 \sum_{n=1}^{\infty} \frac{\lambda}{n+1} z^n \right\} > 0 \quad (|z| < 1).$$

Using the fact that $\operatorname{Re}\{(-1/z)\log(1-z)\} > \log 2$, [7], we have

$$\begin{aligned} \operatorname{Re} \left\{ 1 + 2 \sum_{n=1}^{\infty} \frac{\lambda}{n+1} z^n \right\} &= \operatorname{Re} \left\{ (1-2\lambda) - \frac{2\lambda}{z} \log(1-z) \right\} \\ &> (1-2\lambda) + 2\lambda \log 2 > 0 \end{aligned}$$

if $\lambda < 1/2(1 - \log 2)$.

To show that this result is sharp we consider the function $f(z) = z(1-z)^{-1}$ which is an element of K . We have

$$g(z) = \frac{\lambda}{z} \int_0^z \frac{t}{1-t} dt = \lambda \left\{ -1 - \frac{1}{z} \log(1-z) \right\}.$$

Because $g(-1) = -\lambda\{1 - \log 2\} < -1/2$ if $\lambda > 1/2(1 - \log 2)$ it is, therefore, not true that g is subordinate to f for $|z| < 1$, since the range of f is the half plane $\{w: \operatorname{Re} w > -1/2\}$. The proof of Theorem 4 is, therefore, complete.

Theorem 4 with $\lambda = 1$ was earlier proved by Bernardi [2].

REMARK. If f belongs to S and g is defined as in (8) then, in view of the fact that f is convex in $|z| < 2 - \sqrt{3}$, it follows that the subordination (9) will hold in $|z| < 2 - \sqrt{3}$ for all λ , $0 < \lambda < 1/2(1 - \log 2)$.

THEOREM 5. If f belongs to K and g is defined by

$$g(z) = \mu \int_0^z \frac{f(t)}{t} dt, \quad (11)$$

then for every μ , $0 < \mu < 1/2 \log 2 = 0.721 \dots$, we have

$$g(z) < f(z) \quad (|z| < 1). \quad (12)$$

The result is sharp.

PROOF. In view of Wilf's result the assertion (12) will hold in $|z| < 1$, if

$$\operatorname{Re} \left\{ 1 + 2\mu \sum_{n=1}^{\infty} \frac{1}{n} z^n \right\} > 0 \quad (|z| < 1).$$

Letting $z = re^{i\theta}$, we find that

$$\begin{aligned} \operatorname{Re} \left\{ 1 + 2\mu \sum_{n=1}^{\infty} \frac{1}{n} z^n \right\} &= \operatorname{Re} \{ 1 - 2\mu \log(1-z) \} \\ &= \{ 1 - 2\mu \log(1 + r^2 - 2r \cos \theta)^{1/2} \} \\ &> 1 - 2\mu \log 2 > 0 \end{aligned}$$

provided $\mu < 1/2 \log 2$.

To show that the result is sharp we again consider the function $f(z) = z(1 - z)^{-1}$ which belongs to K . We have

$$g(z) = -\mu \log(1 - z).$$

Thus $g(-1) = -\mu \log 2 < -1/2$, if $\mu > 1/2 \log 2$ and so g is not subordinate to f for $|z| < 1$ since the range of f is the half plane $\{w: \operatorname{Re} w > -1/2\}$. This completes the proof of our theorem.

REMARK. If f belongs to S and g is defined as in (11) then the assertion (12) holds in $|z| < 2 - \sqrt{3}$.

For $f(z) = \sum_{n=1}^{\infty} a_n z^n$, we define

(i) $s_n(z, f) = \sum_{k=1}^n a_k z^k$ and

(ii) $\sigma_n(z, f) = (1/n) \sum_{k=1}^n s_k(z, f)$.

THEOREM 6. For all elements f of K , we have

$$(1/\alpha_n) s_n(z, f) < f(z), \quad |z| < 1, \quad (13)$$

where $\alpha_n = -2 \min_{|z| < 1} \operatorname{Re}\{\sum_{k=1}^n z^k\}$, and

$$(1/\beta_n) \sigma_n(z, f) < f(z), \quad |z| < 1, \quad (14)$$

where $\beta_n = -(2/n) \min_{|z| < 1} \operatorname{Re}\{\sum_{k=1}^n (n - k + 1) z^k\}$.

PROOF. Suppose

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

then

$$s_n(z, f) = z + a_2 z^2 + \cdots + a_n z^n$$

and

$$\sigma_n(z, f) = z + \frac{n-1}{n} a_2 z^2 + \frac{n-2}{n} a_3 z^3 + \cdots + \frac{1}{n} a_n z^n.$$

In view of Wilf's result quoted above we shall have

$$(1/\alpha_n) s_n(z, f) < f(z), \quad |z| < 1,$$

if

$$\operatorname{Re}\left\{1 + \frac{2}{\alpha_n} \sum_{k=1}^n z^k\right\} > 0, \quad |z| < 1,$$

which is true by the definition of α_n .

This completes the proof of (13). Relation (14) may be similarly established.

REMARK. Since $\alpha_n < 2n$ and $\beta_n < n + 1$, $n \geq 1$, relations (13) and (14), in particular, show that

$$(1/2n) s_n(z, f) < f(z), \quad |z| < 1,$$

and

$$(1/(n+1)) \sigma_n(z, f) < f(z), \quad |z| < 1.$$

The well-known result $z/2 < f(z)$ for every f belonging to K corresponds to $n = 1$ in (13) and (14).

It is easy to compute that $\alpha_2 = 9/4$ and $\beta_2 = 3/2$. Thus for $n = 2$ the above theorem gives

$$(4/9)s_2(z, f) < f(z), \quad |z| < 1,$$

and

$$(2/3)\sigma_2(z, f) < f(z), \quad |z| < 1.$$

We will need the following lemma due to Keogh [3] to prove our next theorem.

LEMMA 4. Suppose that b_0, b_1, b_2 are complex numbers, $b_2 \neq 0$, and let $P(z) = b_0 + b_1z + b_2z^2$. Then the zeros of $P(z)$ lie on $|z| < 1$ if, and only if

- (i) $|b_0| \leq |b_2|$ and
- (ii) $|b_0\bar{b}_1 - b_1\bar{b}_2| \leq |b_2|^2 - |b_0|^2$.

THEOREM 7. If f belongs to K , then we have

$$V_2(z, f) < s_2(z, f) \quad (15)$$

in $|z| < 2/5$, and this result is sharp;

$$z/2 < V_2(z, f) < \sigma_2(z, f) \quad (16)$$

in $|z| < 1$.

PROOF. It is well known that $s_2(z, f)$ is starlike univalent in $|z| < 1/2$ and convex in $|z| < 1/4$, and that $V_2(z, f)$ is convex for every convex function f in $|z| < 1$.

First, we want to prove that $V_2(z, f)$ is subordinate to $s_2(z, f)$ in $|z| < 2/5 = r_0$ (say). That is, $V_2(r_0z, f) < s_2(r_0z, f)$ in $|z| < 1$. Following BaşGöze et al. [1], it is sufficient to show that for each real α , the polynomial

$$P(z) = a_2r_0^2z^2 + r_0z - \left(\frac{2}{3}r_0e^{i\alpha} + \frac{1}{6}a_2r_0^2e^{2i\alpha}\right) \quad (17)$$

has a zero on $|z| < 1$. Suppose that for some α it has no zero in $|z| < 1$. Then the polynomial

$$Q(z) = \left(\frac{2}{3}r_0e^{i\alpha} + \frac{1}{6}a_2r_0^2e^{2i\alpha}\right)z^2 - r_0z - a_2r_0^2 \quad (18)$$

has both zeros on $|z| \leq 1$; hence by Lemma 4,

$$\left|a_2r_0^3 + r_0\left(\frac{2}{3}r_0e^{-i\alpha} + \frac{1}{6}\bar{a}_2r_0^2e^{-2i\alpha}\right)\right| \leq \left|\frac{2}{3}r_0e^{i\alpha} + \frac{1}{6}a_2r_0^2e^{2i\alpha}\right|^2 - |a_2r_0^2|^2.$$

Writing $a_2r_0 = \rho e^{i\phi}$, $\alpha + \phi = \psi$, this is equivalent to

$$|4 + \rho e^{i\psi}|^2 - 6|6\rho e^{i\psi} + 4 + \rho e^{-i\psi}| \geq 36\rho^2. \quad (19)$$

One readily verifies that the maximum of the left-hand side of (19) is attained at $\psi = \pi$. Also for $\psi = \pi$ the inequality (i) of Lemma 4 implies that $\rho < 4/7$. Therefore (19) will fail to hold if $\rho^2 + 34\rho - 8 < 36\rho^2$ or

$$(\rho - 2/5)(\rho - 4/7) > 0. \quad (20)$$

Since $\rho < 4/7$, the inequality (19) will not hold for all $\rho < 2/5$. This proves that the polynomial $P(z)$ has all its zeros in $|z| \leq 1$ for $\rho < 2/5$. Since $\rho = |a_2|r_0 \leq r_0$, we have proved that for $r_0 < 2/5$, $P(z)$ has all its zeros in $|z| < 1$ and hence

$$V_2(r_0z, f) < s_2(r_0z, f) \quad \text{in } |z| < 1 \quad (r_0 < 2/5).$$

This establishes the relation

$$V_2(z, f) < s_2(z, f) \quad \text{in } |z| < r_0.$$

To show that our result is sharp in the sense that if $2/5 < r < 1$ then there is a function f in K so that $V_2(z, f)$ is not subordinate to $s_2(z, f)$ for $|z| < r$, we consider the function $f(z) = z/(1 - z)$. Then $V_2(z, f) = (2/3)z + (1/6)z^2$ and $s_2(z, f) = z + z^2$. Since $V_2(-2/5, f) = s_2(-2/5, f)$, $V_2(z, f)$ is not subordinate to $s_2(z, f)$ for $|z| < r$ (if $2/5 < r < 1$). The last assertion depends upon the strict sense in which equality is possible in Lindelöf's Principle for subordination.

As for the proof of (16) we remark that since $\sigma_2(z, f)$ is univalent (in fact starlike) in $|z| < 1$ and the relation $z/2 < V_2(z, f)$ for every f belonging to K being well known, we need to prove only

$$V_2(z, f) < \sigma_2(z, f), \quad |z| < 1.$$

It suffices to show that for each real α , the polynomial

$$R(z) = z + \frac{1}{2}a_2z^2 - \frac{2}{3}e^{i\alpha} - \frac{1}{6}a_2e^{2i\alpha} \quad (21)$$

has a zero on $|z| < 1$. Suppose that for some α it has no zero in $|z| < 1$. Then the polynomial

$$T(z) = \left(\frac{2}{3}e^{i\alpha} + \frac{1}{6}a_2e^{2i\alpha} \right) z^2 - z - \frac{1}{2}a_2 \quad (22)$$

has both zeros on $|z| < 1$; hence by Lemma 5,

$$\left| \frac{1}{2}a_2 + \frac{2}{3}e^{-i\alpha} + \frac{1}{6}\bar{a}_2e^{-2i\alpha} \right| < \left| \frac{2}{3}e^{i\alpha} + \frac{1}{6}a_2e^{2i\alpha} \right|^2 - \left| \frac{1}{2}a_2 \right|^2.$$

Writing $a_2 = \rho e^{i\phi}$, $\alpha + \phi = \delta$, this is equivalent to

$$|4 + \rho e^{i\delta}|^2 - 6|3\rho e^{i\delta} + 4 + \rho e^{-i\delta}| > 9\rho^2. \quad (23)$$

Proceeding as in the proof of (15) we arrive at the conclusion that for $\rho < 1$, (23) is not true and hence $R(z)$ will have all its zeros in $|z| < 1$ for $\rho < 1$. This then will complete the proof of (16).

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