PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 1, May 1981

A CONTINUOUS VERSION OF THE BORSUK-ULAM THEOREM

JAN JAWOROWSKI[1]

ABSTRACT. Let p: $E \to B$ be an n-sphere bundle, q: $V \to B$ be an $\mathbf{R}^n$-bundle and f: $E \to V$ be a fibre preserving map over a paracompact space B. Let $\bar{p}$: $\bar{E} \to B$ be the projectivized bundle obtained from p by the antipodal identification and let $\bar{A}_f$ be the subset of $\bar{E}$ consisting of pairs $\{e, -e\}$ such that $fe = f(-e)$. If the cohomology dimension d of B is finite then the map $(\bar{p}|\bar{A}_f)^*$: $H^d(B; \mathbf{Z}_2) \to H^d(\bar{A}_f; \mathbf{Z}_2)$ is injective for a continuous cohomology theory H^*. Moreover, if the jth Stiefel-Whitney class of q is zero for $1 < j < r$ then $(\bar{p}|\bar{A}_f)^*$ is injective in degrees $i > d - r$. If all the Stiefel-Whitney classes of q are zero then $(\bar{p}|\bar{A}_f)^*$ is injective in every degree.

Introduction. The Borsuk-Ulam theorem [1] says that if f: $S^n \to \mathbf{R}^n$ is a map then the set A_f of points $x \in S^n$ such that $fx = f(-x)$ is nonempty. Because A_f is symmetric with respect to the antipodal involution, it is more convenient to consider the subset $\bar{A}_f$ of the real projective n-space P^n corresponding to A_f under the antipodal identification.

If a single S^n and an $\mathbf{R}^n$ are replaced by continuous families $E \to B$ with fibre S^n and $V \to B$ with fibre $\mathbf{R}^n$ over a space B, and if f is replaced by a fibre preserving map f: $E \to V$, one may expect the existence of a cross-section of sorts in the set $\bar{A}_f$ of pairs $\{e, -e\}$ such that $e \in E$ and $fe = f(-e)$, at least on an algebraic level.

A result in this direction in the case when E is the product bundle $E = S^k \times S^n$ and V is a single $\mathbf{R}^n$ follows from a theorem proved by J. E. Connett [2]. In this note we are going to consider this question for fibre preserving maps $E \to V$ where E is an n-sphere bundle and V is an n-dimensional real vector space bundle over a paracompact space B. If B is a point, then the theorem proved below reduces to the classical Borsuk-Ulam theorem.

Main result. If X is a space with an involution t: $X \to X$, we denote by $\bar{X}$ the orbit space X/t of t. If p: $E \to B$ is a fibre bundle with a fibre preserving involution t: $E \to E$, we write $\bar{p}$: $\bar{E} \to B$ for the bundle p/t: $E/t \to B$; its fibre is $\bar{X}$, where X is the fibre of p. Thus if p: $E \to B$ is an n-sphere bundle, then $\bar{p}$: $\bar{E} \to B$ is the associated real projective n-space bundle.

Received by the editors June 30, 1980; presented at the 775th meeting of the American Mathematical Society, Special Session on Fixed Point Theory, in Bloomington, Indiana, on April 12, 1980.

1980 *Mathematics Subject Classification*. Primary 55R25, 55R20; Secondary 55M35.

Key words and phrases. Sphere bundle, vector space bundle, antipodal map, involution, fibre preserving map, Stiefel-Whitney class.

[1]The author gratefully acknowledges the support received from the Forschungsinstitut Für Mathematik, ETH Zürich, during the preparation of this paper. He is also grateful to Albrecht Dold for helpful conversations.

If E is any space with an involution t: $E \to E$ and f: $E \to V$ is a map of E into some space V, let A_f denote the set of points $e \in E$ such that $fe = fte$ and let $\bar{A}_f$ be the image of A_f in $\bar{E}$.

We are going to use the Alexander-Spanier cohomology theory H^* mod 2. The coefficient group $\mathbf{Z}_2$ will be suppressed from the notation. If Z is a space, A is a subset of Z and i: $A \to Z$ is the inclusion map, then the image of a cohomology class $z \in H^*(Z)$ under the induced homomorphism i^*: $H^*(Z) \to H^*(A)$ will sometimes be denoted by $z|A$ and called the restriction of z to A. We denote by dim Z the covering dimension of Z and by $d(Z)$ its cohomology dimension, that is, $d(Z) = \mathrm{Sup}\{m\colon H^m(Z) \neq 0\}$. We have $d(Z) \leqslant \dim Z$ if Z is paracompact. If q: $V \to B$ is a vector space bundle over B then the jth Stiefel-Whitney class of q is denoted by $w_j(q)$.

We will assume throughtout the paper that B is a paracompact space.

THEOREM. *Let $p: E \to B$ be an n-sphere bundle with the antipodal involution, let q: $V \to B$ be an $\mathbf{R}^n$-bundle and let f: $E \to V$ be a fibre preserving map over B. If $d(B) \leqslant d$ and $w_j(q) = 0$ for $1 \leqslant j \leqslant r$ then the map $(\bar{p}|\bar{A}_f)^*$: $H^i(B) \to H^i(\bar{A}_f)$ is injective for $i \geqslant d - r$.*

In the following corollaries we specify particular cases of this theorem to illustrate its significance.

COROLLARY 1. *If f: $E \to V$ is a fibre preserving map of an n-sphere bundle p: $E \to B$ with the antipodal involution into an $\mathbf{R}^n$-bundle q: $V \to B$ and if $d(B) = d < \infty$, then the map $(\bar{p}|\bar{A}_f)^*$: $H^d(B) \to H^d(\bar{A}_f)$ is injective.*

COROLLARY 2. *If f: $E \to V$ is a fibre preserving map of an n-sphere bundle p: $E \to B$ with the antipodal involution into an $\mathbf{R}^n$-bundle q: $V \to B$ and if all the Stiefel-Whitney classes of q are zero then the map $(\bar{p}|\bar{A}_f)^*$: $H^i(B) \to H^i(\bar{A}_f)$ is injective for every i.*

COROLLARY 3. *If B is closed manifold and f: $E \to V$ is a fibre preserving map of an n-sphere bundle p: $E \to B$ with the antipodal involtion into an $\mathbf{R}^n$-bundle q: $V \to B$ then* $\dim A_f = \dim \bar{A}_f \geqslant \dim B$.

In Corollary 3, we have $d = d(B) = \dim B$ and $H^d(B) \neq 0$. On the other hand, $\dim \bar{A}_f = \dim A_f$ since the orbit map $A_f \to \bar{A}_f$ is a double covering.

Proof of the theorem. If X is any space with a free involution t: $X \to X$, let $u(X)$ denote its characteristic class. It is an element $u(X) \in H^1(\bar{X})$, where $\bar{X}$ is, as usual, the orbit space of t. In other words, $u(X)$ is the Stiefel-Whitney class of the double covering $X \to \bar{X}$. The class $u(S^n)$ of the antipodal involution generates the polynomial ring $H^*(P^n)$ of height n.

Let $b \in B$. Then the fibre of $\bar{p}$ over b is $\bar{p}^{-1}b \cong P^n$ and the polynomial ring $H^*(\bar{p}^{-1}b)$ is generated by $u(p^{-1}b) \in H^1(\bar{p}^{-1}b)$. The fibre inclusion $p^{-1}b \to E$ is an equivariant map. By the naturality of u, the restriction of $u(E) \in H^1(\bar{E})$ to the fibre $\bar{p}^{-1}b$ is equal to $u(p^{-1}b)$. By the Leray-Dold-Hirsch theorem [3, p. 229], $H^*(\bar{E})$ is an $H^*(B)$-module freely generated by the powers 1, $u(E), \ldots, u^n(E)$, with

$H^*(B)$ acting on $H^*(\bar{E})$ via the cup product. In other words, the map

$$\bigoplus_{i=0}^{n} H^{m+i}(B) \to H^{m+n}(\bar{E}),$$

$$(x_m, x_{m+1}, \ldots, x_{m+n}) \mapsto \sum_{i=0}^{n} (\bar{p}^* x_{m+i}) \cup u^{n-i}(E)$$

is an isomorphism. This map restricted to $H^m(B)$ gives a monomorphism

$$\iota\colon H^m(B) \to H^{m+n}(\bar{E}), \qquad x \mapsto (\bar{p}^* x) \cup u^n(E).$$

Let 0 be the zero section in V and $V_0 = V - 0$. Then the antipodal map is a free involution in V_0 and the fibre of the bundle $q_0 = q|V_0\colon V_0 \to B$ is $\mathbf{R}_0^n = \mathbf{R}^n - (0)$. The bundle q_0 is fibre homotopy equivalent to its S^{n-1}-bundle and hence $H^*(\bar{V}_0)$ is an $H^*(B)$-module freely generated by 1, $u(V_0), \ldots, u^{n-1}(V_0)$. Moreover, $u^n(V_0) = \sum_{j=1}^n (\bar{q}_0^* w_j) \cup u^{n-j}(V_0)$, where the coefficient $w_j = w_j(q)$ is the jth Stiefel-Whitney class of q [3, p. 232].

Let $g\colon E \to V$ be defined by $ge = fe - f(-e)$. Then g is equivariant, $g(-e) = -ge$, $A_f = A_g = g^{-1}0$ and the restriction of g to $E_0 = E - A_f$ defines an equivariant map $g_0\colon E_0 \to V_0$. By the naturality of u, we have $\bar{g}_0^* u(V_0) = u(E_0)$, where $\bar{g}_0\colon \bar{E}_0 \to \bar{V}_0$ is the map of the orbit bundles induced by g_0, and $u(E_0) = u(E)|\bar{E}_0$. It follows that

$$\begin{aligned} u^n(E)|\bar{E}_0 = \bar{g}_0^* u^n(V_0) &= \sum_{j=1}^{n} \left[(\bar{p}^* w_j)|\bar{E}_0\right] \cup \left[u^{n-j}(E)|\bar{E}_0\right] \\ &= \left[\sum_{j=1}^{n} (\bar{p}^* w_j) \cup u^{n-j}(E)\right]\Big|\bar{E}_0. \end{aligned}$$

To show that $(\bar{p}|\bar{A}_f)^*$ is a monomorphism in the degrees specified in the theorem, suppose that $x \in H^i(B)$ with $i \geqslant d - r$ and $(\bar{p}|\bar{A}_f)^* x = 0$, i.e., $(\bar{p}^* x)|\bar{A}_f = 0$. By the continuity of H^*, there is a neighborhood U of A_f in E such that $(\bar{p}^* x)|\bar{U} = 0$ ($\bar{U}$ denotes, as usual, the image of U in $\bar{E}$). Let $e\colon \bar{E} \to (\bar{E}, \bar{U})$ and $k\colon \bar{E} \to (\bar{E}, \bar{E}_0)$ be the inclusion maps. Since $(\bar{p}^* x)|\bar{U} = 0$, then $\bar{p}^* x = e^* y$, for some $y \in H^i(\bar{E}, \bar{U})$. Let $v = u^n(E) - \sum_{j=1}^n (\bar{p}^* w_j) \cup u^{n-j}(E)$. Then $v|\bar{E}_0 = 0$; hence $v = k^* z$, for some $z \in H^n(\bar{E}, \bar{E}_0)$. Since $(\bar{E}; \bar{U}, \bar{E}_0)$ is an excisive triad, $e^* y \cup k^* z = y \cup z = 0$; hence $0 = (\bar{p}^* x) \cup u^n(E) - (\bar{p}^* x) \cup [\sum_{j=1}^n (\bar{p}^* w_j) \cup u^{n-j}(E)]$. Therefore

$$(\bar{p}^* x) \cup u^n(E) = \sum_{j=1}^{n} \bar{p}^*(x \cup w_j) \cup u^{n-j}(E).$$

Now if $j \leqslant r$ then $w_j = 0$ by the assumption. If $j > r$ then $\deg(x \cup w_j) = i + j > i + r \geqslant d \geqslant d(B)$ since $i \geqslant d - r$. Therefore all the coefficients in this polynomial are zero. Hence $(\bar{p}^* x) \cup u^n(E) = 0$. But $(\bar{p}^* x) \cup u^n(E) = \iota x$ and ι is a monomorphism. Therefore $x = 0$ and thus $(\bar{p}|\bar{A}_f)^*$ is a monomorphism. Q.E.D.

References

1. K. Borsuk, *Drei Sätze über die n-dimensionale Euklidische Sphäre*, Fund. Math. **20** (1933), 177–190.
2. J. E. Connett, *On the cohomology of the fixed-point sets and coincidence-point sets*, Indiana Univ. Math. J. **24** (1974–75), 627–634.
3. D. Husemoller, *Fibre bundles*, McGraw-Hill, New York, 1966.

Forschungsinstitut für Mathematik, ETH Zürich, Zurich, Switzerland

Department of Mathematics, Indiana University, Bloomington, Indiana 47401 (Current address)