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ORIENTABILITY OF FIXED POINT SETS

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ABSTRACT. It is proved that the fixed point set of a smooth involution which preserves orientation and a spin structure on a smooth manifold is necessarily orientable. As an application it is proved that a simply connected spin 4-manifold with nonzero signature admits no involution which acts by multiplication by -1 on its second rational homology group.

Consider a smooth orientation-preserving action of the cyclic group of order 2, Z_2, on a manifold M. As is well known, in contrast to the case of odd order group actions, the fixed point set F is a manifold which need not be orientable. The simplest example is that of complex conjugation on $\mathbf{C}P^2$ which fixes $\mathbf{R}P^2$. In this note we shall show that if in addition the action preserves a spin structure on M, then F is necessarily orientable. This result will then be applied to show that a closed simply connected spin 4-manifold M^4 of nonzero signature does not support a smooth Z_2 action which acts by multiplication by -1 on $H_2(M; Q)$.

The crux of the argument is in the following proposition about Z_2 vector bundles over the 2-sphere.

PROPOSITION 1. *Let ξ be a k-dimensional Z_2 vector bundle over S^2, where Z_2 acts on S^2 by reflection through the equator S^1. Then the fixed subbundle $(\xi|S^1)^{Z_2}$ is orientable if and only if ξ is stably trivial as a vector bundle.*

PROOF. If $\dim(\xi|S^1)^{Z_2} = \dim(\xi|S^1)$, then any nonequivariant trivialization of $\xi|D^2_+$, where D^2_+ is the upper hemisphere of S^2, extends to an equivariant trivialization of ξ. So in this case both ξ and $(\xi|S^1)^{Z_2}$ are trivial.

Next consider the case when ξ is 2-dimensional and $(\xi|S^1)^{Z_2}$ is 1-dimensional. Let $S^2 \subset E(\xi)$ denote the zero section. Then ξ is stably trivial if and only if the self-intersection number $S^2 \cdot S^2 \equiv 0 \pmod 2$. To compute $S^2 \cdot S^2$ we carefully perturb S^2 to a section $\bar{S}^2$ transverse to S^2 and count $S^2 \cap \bar{S}^2$.

Suppose that $(\xi|S^1)^{Z_2}$ is trivial. First perturb the equator S^1 in $E(\xi|S^1)^{Z_2}$ to a nonvanishing section $\bar{S}^1$. This extends to some section $\bar{D}^2_+$ of $\xi|D^2_+$, transverse to D^2_+. By equivariance this extends to a section $\bar{D}^2_-$ of $\xi|D^2_-$, transverse to D^2_-. Then $\bar{S}^2 = \bar{D}^2_+ \cup \bar{D}^2_-$ is a topological section transverse to S^2. Clearly $\bar{S}^2 \cap S^2$ consists of an even number of points.

Now suppose $(\xi|S^1)^{Z_2}$ is nontrivial. Then we may choose a section $\bar{S}^1$ of $(\xi|S^1)^{Z_2}$ which vanishes at exactly one point of transverse intersection with the zero section S^1. This section extends to a section $\bar{D}^2_+$ of $\xi|D^2_+$ whose interior meets D^2_+ in

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finitely many points of transverse intersection. By equivariance extend this to a topological section $\bar{S}^2$ of ξ. Clearly $\bar{S}^2 \cap S^2$ consists of an odd number of points. It remains only to observe that $\bar{S}^2$ is topologically transverse to S^2. To see this consider a standard model for a neighborhood of $\{x\} = S^1 \cap \bar{S}^1$ in $E(\xi)$, as follows: Coordinatize a neighborhood by $\mathbf{C} \times \mathbf{C}$, where the first factor is the base and the second factor is the fiber, and Z_2 acts by complex conjugation in each factor. Then S^1 corresponds to $\mathbf{R} \times 0$ and $(\xi|S^1)^{Z_2}$ corresponds to $\mathbf{R} \times \mathbf{R}$. Simply perturb $\mathbf{R} \times 0$ to $\bar{\mathbf{R}} = \{(x, x)\colon x \in \mathbf{R}\}$. This may be accomplished by perturbing $\mathbf{C} \times 0$ to $\bar{\mathbf{C}} = \{(z, z)\colon z \in \mathbf{C}\}$. Then clearly $\bar{\mathbf{R}}$ is transverse to $\mathbf{R} \times 0$ in $\mathbf{R} \times \mathbf{R}$ and $\bar{\mathbf{C}}$ is transverse to $\mathbf{C} \times 0$ in $\mathbf{C} \times \mathbf{C}$. Since the perturbation $\bar{S}^2$ of S^2 above may be chosen this way near $\bar{S}^1 \cap S^1$, we see that $\bar{S}^2$ is indeed transverse to S^2 as claimed.

It remains to reduce the general case to the cases already considered. First of all, we may assume $(\xi|S^1)^{Z_2}$ has positive dimension by adding to ξ a trivial bundle $S^2 \times \mathbf{R}$ with trivial Z_2 action in the fiber, if necessary. Clearly this does not affect the orientability of $(\xi|S^1)^{Z_2}$ or the stable triviality of ξ.

Decompose $\xi|S^1 = \xi^r_{-1} \oplus \xi^s_{+1}$ into eigenbundles for the eigenvalues ± 1, where $r + s = k$. We have arranged that $s \geqslant 1$. The first case considered was when $r = 0$; so we may also assume $r \geqslant 1$. The two eigenbundles can be destabilized to line bundles:

$$\xi^r_{-1} \approx \varepsilon^{r-1}_{-1} \oplus \xi^1_{-1} \quad \text{and} \quad \xi^s_{+1} \approx \varepsilon^{s-1}_{+1} \oplus \xi^1_{+1}.$$

Let $\xi^2_0 = \xi^1_{-1} \oplus \xi^1_{+1}$ and $\varepsilon = \varepsilon^{r-1}_{-1} \oplus \varepsilon^{s-1}_{+1}$. We claim that the equivariant destabilization $\xi|S^1 = \xi^2_0 \oplus \varepsilon^{k-2}$ extends to an equivariant destabilization of ξ over S^2, $\xi^2 \oplus \varepsilon$.

To accomplish this extension it suffices to show that an invariant trivial line bundle $\lambda \subseteq \xi|S^1$ extends to a (necessarily trivial) invariant line bundle $\mu \subseteq \xi$ over S^2. To do this, choose a nonvanishing section of $\xi|S^1$ lying in λ. This section extends to a nonvanishing section of $\xi|D^2_+$, since the obstruction to such an extension lies in $\pi_1(\mathbf{R}^k - \{0\})$, which is 0 since $k \geqslant 3$. This determines a line bundle $\mu_+ \subset \xi|D^2_+$ extending λ. Then $\mu_- = T(\mu_+)$ is a line bundle over D^2_-, which extends λ since T preserves λ. Thus $\mu_+ \cup \mu_-$ is the desired line bundle.

Finally ξ is stably trivial if and only if ξ^2 is; and $(\xi|S^1)^{Z_2}$ is orientable if and only if $(\xi^2|S^1)^{Z_2}$ is. Reference to the second case considered completes the proof.

For subsequent use we note that Proposition 1 also holds, by the same proof, for orientable vector bundles over any closed surface on which Z_2 acts by reflection through a separating simple closed curve.

An alternative proof of half of Proposition 1 assuming ξ is a trivial vector bundle over S^2 might go as follows: Equivalence classes of Z_2 vector bundle structures on $S^2 \times \mathbf{R}^k$ correspond bijectively with the set of homotopy classes of maps $[D^2, S^1; O_k, O_k^{Z_2}]$, where Z_2 acts on the orthogonal group O_k by inversion (see Bredon [3, VI.11.1]). The triviality of $(\xi|S^1)^{Z_2}$ in this case is the equivalent to saying $\pi_2(O_k, O_k^{Z_2}) = 0$ for all choices of base point–i.e., that the above set of homotopy classes corresponds bijectively with the components of $O_k^{Z_2}$, a collection of Grassmann manifolds.

In order to apply this we need a short discussion of spin structures (see Atiyah-Bott [1] and Milnor [7]). A *spin* structure on an oriented manifold M is a (stable) reduction of its tangent bundle τ_M from the special orthogonal group SO to its double covering Spin. The only obstruction to the existence of a spin structure is the second Stiefel-Whitney class $w_2(M) \in H^2(M; Z_2)$. If $w_2(M) = 0$, the distinct spin structures are in one-to-one correspondence with $H^1(M; Z_2)$. Alternatively a spin structure on M is an element $\omega \in H^1(FM; Z_2)$, where FM is the tangent frame bundle of M, such that ω restricts nontrivially to the fiber SO. Then a diffeomorphism *preserves* ω if $(dT)^*(\omega) = \omega$.

More geometrically, a spin structure ω on M may be viewed as a framing $\mathcal{F}$, or trivialization, of the restriction of the tangent bundle τ_M to a neighborhood of the 2-skeleton of some triangulation of M. And a diffeomorphism preserves ω if it preserves the framing $\mathcal{F}$.

In applications of Proposition 1 the following lemma will allow us to assume our manifold M is 1-connected.

LEMMA 2. *Let Z_2 act smoothly, preserving orientation and a spin structure on an n-manifold M^n, $n \geqslant 4$, with fixed point set F. Then there is a cobordism W^{n+1} from M^n to a manifold N^n on which Z_2 acts preserving a spin structure extending the spin structure on M^n with the properties that*

(1) $\pi_1(N^n) = 0$ *and*

(2) $\mathrm{Fix}(Z_2, W) \cong F \times [0, 1]$.

PROOF. Let T: $M^n \to M^n$ be the generator of the Z_2-action. Let $\{x_i\} \subset \pi_1(M^n)$ be a finite set of (normal) generators for $\pi_1(M^n)$. By general position, these elements x_i can be represented by disjoint embedded circles $C_i \subset M^n$ such that $T(C_i) \cap C_j = \varnothing$ for all i, j.

As in Milnor [6, Theorems 2, 3], and, especially, Kervaire and Milnor [5, Lemma 6.2], there are *unique* framings $\mathcal{F}_i$ of the normal bundles $\nu(C_i, M^n)$ so that the given spin structure extends over the trace of surgery on $\{C_i\}$, using the $\{\mathcal{F}_i\}$. Similarly there are unique framings $\mathcal{G}_i$ of $\nu(TC_i, M^n)$ such that the given spin structure extends over handles attached to $M \times I$ along these framed circles. But since T preserves the given spin structure, it follows that $\mathcal{G}_i = T^*\mathcal{F}_i$. From this, it then follows that one can surger $\{C_i\} \cup \{TC_i\}$ simultaneously and equivariantly using framings $\{\mathcal{F}_i\} \cup \{T^*\mathcal{F}_i\}$, and that the trace of the surgery has the desired properties.

An instructive example is a Z_2 action on $S^1 \times S^3$ fixing a Klein bottle: simply double a regular neighborhood of a nontrivial circle in $\mathbf{R}P^2 \subset \mathbf{C}P^2$, invariant under complex conjugation. In this case the Z_2 action switches the two spin structures on $S^1 \times S^3$. Equivariant surgery to kill the fundamental group leads to $\mathbf{C}P^2 \# -\mathbf{C}P^2$, rather than $S^2 \times S^2$.

THEOREM 3. *If Z_2 acts smoothly on an n-manifold M^n, preserving orientation and a spin structure, then the fixed point set $F = M^{Z_2}$ is orientable.*

PROOF. We may assume that M is connected and Z_2 acts nontrivially. Since the action preserves orientation, M^{Z_2} has even positive codimension. Therefore we need only consider the case when $n > 4$. By Lemma 2 we may also assume M is simply connected. Suppose F is nonorientable. Then there is an orientation-reversing loop f_0: $S^1 \to F$. Since $\pi_1(M) = 0$, f_0 extends to a map f_+: $D^2_+ \to M$. By equivariance f_+ extends to an equivariant map f: $S^2 \to M$, where, as usual, Z_2 acts on S^2 by reflection through S^1. Define $\xi = f^*(\tau_M)$, where τ_M is the tangent bundle of M. Since $f_0(S_1)$ is orientation-reversing, $(\xi|S^1)^{Z_2} = f_0(\tau_M|F)^{Z_2} = f_0^*(\tau_F)$ is nontrivial. By Proposition 1, ξ is stably nontrivial. On the other hand $w_2(\xi) = f^* w_2(\tau_M) = f^*(0) = 0$, since M is a spin manifold, implying that ξ is stably trivial. This contradiction completes the proof.

The same line of proof applies to the following consequence of Proposition 1, which concerns involutions which do not necessarily preserve a spin structure.

COROLLARY 4. *Let Z_2 act smoothly on a spin manifold M with fixed point set F. Then the orientation class $w_1(F)$ lies in the image of the restriction homomorphism $H^1(M; Z_2) \to H^1(F; Z_2)$.*

PROOF. If $w_1(F)$ is not in the image of the restriction homomorphism, then there is an orientation-reversing loop f_0: $S^1 \to F$ which bounds in M mod 2. So F_0 extends to a map f_+: $V^2_+ \to M$ for some possibly nonorientable surface V^2_+ with boundary S^1. This yields an equivariant map f: $V^2 \to M$ where $V^2 = V_+ \cup_{S^1} V_-$ and V_- is another copy of V_+, and Z_2 acts on V by reflection through S^1.

Now the proof proceeds just as that of Theorem 3, applying the version of Proposition 1 described at the conclusion of the proof of the proposition.

As an application of the preceding results we have the following theorem which was the original motivation of this work.

THEOREM 5. *Let Z_2 act smoothly on an orientable closed smooth spin 4-manifold M preserving orientation and a spin structure. Suppose $H_2(M; Q)^{Z_2} = 0$. Then M has signature $\sigma M = 0$.*

PROOF. By Lemma 2 we may assume that M is simply connected and hence has a unique spin structure. By a result[2] of Atiyah and Bott [**1**, 5.7] (see also Bredon [**2**, Theorem III]), the fixed point set F, if nonempty, has constant dimension 0 or 2. Suppose first that F consists of isolated points. Then $\chi(M) = 2\chi(M/Z_2) - \# F$. But $H_*(M/Z_2; Q) \approx H_*(M; Q)^{Z_2} \approx H_*(S^4; Q)$, so $2 + \dim H_2(M; Q) = 4 - \# F$. Therefore $\dim H_2(M; Q) < 2$; and $\sigma(M) = 0$ since a spin 4-manifold of nonzero signature has $\dim H_2(M; Q)$ at least 8 (in fact 16, by Rochlin's Theorem).

Now consider the case when each component of F has dimension 2. Then the orbit space M/Z_2 is a manifold and the orbit map $M \to M/Z_2$ can be viewed as a smooth branched covering branched over the smooth surface $F \subset M/Z_2$. The surface F is orientable by Theorem 3. As before M/Z_2 is a rational homology sphere.

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[2]As Atiyah has pointed out in a private communication, the italicized statement on p. 487 of [1] preceding Proposition 5.7 should be modified in the nonsimply connected case to say that a fixed point free involution preserving a spin structure ω on a manifold M has even type wth respect to ω if and only if ω is induced from a spin structure on the orbit space. In particular the antipodal involution on S^1 has odd type with respect to the spin structure which extends over a disk and even type with respect to the other one.

The normal bundle $\nu(F, M/Z_2)$ is classified by its Euler class $e(\nu)$ in $H^2(F; Z)$, which is torsion free. But the Euler class must be the restriction of a class in $H^2(M/Z_2; Z)$, which is finite. Therefore $e(\nu) = 0$ and $\nu(F, M/Z_2)$ is trivial. Since $\nu(F, M/Z_2)$ is trivial a specialization of the Atiyah-Bott-Singer fixed point theorem (cf. Hirzebruch [4, p. 255]), says that $\sigma(M) = 2\sigma(M/Z_2)$. Hence $\sigma(M) = 0$.

Finally, if $F = \varnothing$, then, just as in the preceding case, $\sigma(M) = 2\sigma(M/Z_2)$; and $\sigma(M/Z_2) = 0$, since $H_2(M/Z_2; Q) \approx H_2(M; Q)^{Z_2} = 0$. This completes the proof.

Theorem 5 should be viewed as a nonrealizability statement about equivariant bilinear forms. For example, no complex hypersurface of even degree greater than 2 in $\mathbf{C}P^3$ admits an involution which acts on its middle homology by multiplication by -1. In particular no simply connected closed spin 4-manifold of nonzero signature can be a 2-fold branched covering of the 4-sphere.

References

1. M. F. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes*. II. *Applications*, Ann. of Math. **88** (1968), 451–491.

2. G. E. Bredon, *Representations at fixed points of smooth actions of compact groups*, Ann. of Math. **89** (1969), 515–532.

3. ______, *Introduction to compact transformation groups*, Academic Press, New York, 1972.

4. F. Hirzebruch, *The signatures of ramified coverings*, Global Analysis, University of Tokyo Press, Tokyo, 1969, pp. 253–265.

5. M. A. Kervaire and J. W. Milnor, *Groups of homotopy spheres*. I, Ann. of Math. **77** (1963), 504–537.

6. J. W. Milnor, *A procedure for killing homotopy groups of differentiable manifolds*, Proc. Sympos. Pure Math., vol. 3, Amer. Math. Soc., Providence, R. I., 1961, pp. 39–55.

7. ______, *Spin structures on manifolds*, L'Enseignement Math. **9** (1963), 198–203.

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