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AN UNKNOTTING THEOREM IN Q^∞-MANIFOLDS

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ABSTRACT. In this note, we prove the following unknotting theorem.

THEOREM. *Let M be a Q^∞-manifold and let F: $X \times I \to M$ be a homotopy such that F_0 and F_1 are Q^∞-deficient embeddings. Then, there is an isotopy H: $M \times I \to M$ such that $H_0 = \mathrm{id}$ and $H_1 \circ F_0 = F_1$. Moreover, if F is limited by an open cover α of M and is stationary on a closed subset X_0 of X, then we may choose H to also be limited by* $\mathrm{St}^4(\alpha)$ *and to be the identity on $F(X_0 \times I)$.*

However, a similar unknotting theorem for Z-embeddings does not hold true in Q^∞ and R^∞.

1. Introduction and definitions. Given an open cover α of a topological space M, two maps f, g: $X \to M$ are said to be *α-close* if for each $x \in X$, there is an open set $U \in \alpha$ such that $f(x)$, $g(x) \in U$. A homotopy H: $X \times I \to M$ is said to be *α-limited* if for each $x \in X$, there is an open set $U \in \alpha$ such that $H(x \times I) \subset U$.

A closed subset X of M is said to be a *Z-set* in M provided that for any prechosen open cover α of M, there is a map f: $M \to M - X$ that is α-close to id_M.

The *Hilbert cube Q* is the countably infinite product $\prod_1^\infty[0, 1]$. For basic results in Q-manifold theory, we refer to [**1**].

Let $\{Q_n\}$ be a sequence of copies of the Hilbert cube such that Q_n is a Z-set in Q_{n+1} for each n. We define Q^∞ to be the direct limit space $\varinjlim\{Q_n\}$ endowed with the direct limit topology. It follows from the Z-set unknotting theorem of Anderson, Lemma C below, that this space is unique up to homeomorphism. By a *Q^∞-manifold* we mean a paracompact space which is locally homeomorphic to Q^∞.

In [**3**], Heisey proved the following lemma.

LEMMA A. *Every Q^∞-manifold M is equal to $\varinjlim M_n = \bigcup_1^\infty M_n$, where each M_n is a compact Q-manifold, and a Z-set in M_{n+1}.*

From this lemma we may introduce the following notion. A subset X of a Q^∞-manifold M is said to be an *inductive Z-set* in M provided that there is a decomposition $M = \varinjlim M_j$ as in Lemma A such that $X \cap M_j$ is a Z-set in M_j for each $j = 1, 2, \ldots$.

As in [**5**], a closed subset X of a Q^∞-manifold M is said to be *Q^∞-deficient* if there is a homeomorphism h: $M \to M \times Q^\infty$ such that $h(X) \subset M \times 0$. The

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concept of Q^∞-deficiency is equivalent to that of Q-deficiency in the Q^∞-manifold theory [5].

We now state some known results that we will use extensively in the sequel.

LEMMA B [1, THEOREM 3.1]. *Let X and Y be locally compact, separable, metric spaces.*

(i) *If $A \subset X$ is closed and $A = \bigcup_1^\infty A_n$, where $A_n \subset X$ is a Z-set, then $A \subset X$ is a Z-set.*

(ii) *If $A \subset U \subset X$, $A \subset X$ closed, U open and $A \subset U$ is a Z-set, then $A \subset X$ is a Z-set.*

(iii) *If $A \subset X$ is a Z-set, X is an* ANR *and $U \subset X$ is open, then $A \cap U$ is a Z-set in U.*

LEMMA C [1, THEOREM 19.4]. *Let M be a Q-manifold, A be locally compact and let: F: $A \times I \to M$ be a proper map such that F_0 and F_1 are Z-embeddings. Then, there is an isotopy H: $M \times I \to M$ such that $H_0 = \mathrm{id}$ and $H_1 \circ F_0 = F_1$. Moreover, if F is limited by an open cover α of M, then we may choose H to also be limited by α.*

The Main Theorem of this note is the controlled relative version of the unknotting theorem for Q^∞-deficient embeddings in Q^∞-manifolds, which is similar to Lemma C above. Besides that, it will be shown that the notion of Q^∞-deficiency is equivalent to that of inductive Z-embedding in Q^∞-manifolds.

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2. Inductive Z-embedding and Q^∞-deficiency.

PROPOSITION 1. *A subset X of a Q^∞-manifold N is Q^∞-deficient in N if and only if X is an inductive Z-set in N.*

PROOF. (i) Let X be a Q^∞-deficient subset of N. We will show that X is an inductive Z-set in N. We may assume $X \subset N \times 0 \subset N \times Q$. Let $N = K \times Q^\infty$, where K is a locally finite complex (see [3]), and let $K = \bigcup_1^\infty K_n$, where K_n is a compact subcomplex of K_{n+1} for each n. Now, let N_n denote $K_n \times Q_n \times Q$, then $X \cap N_n \subset K_n \times Q_n \times 0$ is a Z-set in N_n. Moreover, it is clear that $N \times Q = \varinjlim (N_n, i_n)$.

(ii) Let X be an inductive Z-set in N. We will construct a homeomorphism h: $N \to N \times Q$ such that $h(X) \subset N \times 0$. Assume that $N = \varinjlim N_n$ such that $X_n = X \cap N_n$ is a Z-set in N_n. We will construct a sequence of homeomorphisms h_n: $N_n \to N_n \times Q$ such that for each n

(a) $h_n(X_n) \subset N_n \times 0$, and

(b) $h_{n+1}|N_n = h_n$.

Then, the homeomorphism h that we desired will be $\varinjlim h_n$, and the proof will be complete.

1. *Construction of* h_1. From Lemma C, there is a homeomorphism f_1: $N_1 \to N_1 \times Q$ such that $f_1(X_1) \subset N_1 \times 0$. Since the projection π: $N_1 \times Q \to N_1 \times 0$ is a

near-homeomorphism, so is the map πf_1. Thus, πf_1 is homotopic to a homeomorphism g_1: $N_1 \to N_1 \times 0$. Let us define $h_1 = (g_1^{-1} \times \mathrm{id}_Q) \circ f_1$: $N_1 \to N_1 \times Q$. Then h_1 is a homeomorphism enjoying the following properties:

(α_1) $h_1(X_1) \subset N_1 \times 0$, and

(β_1) $h_1 \simeq$ inclusion i, where $i(x) = (x, 0)$.

2. *Construction of h_2.* Recall that N_1 is a Z-set in N_2. Let i_1: $N_1 \to N_2$ denote the inclusion map. Similar to the construction of h_1, there is a homeomorphism h_2': $N_2 \to N_2 \times Q$ such that

(α_2) $h_2'(X_2) \subset N_2 \times 0$, and

(β_2) $h_2' \simeq$ inclusion i, $i(x) = (x, 0)$.

Then it can be shown that the Z-embedding $h_2' \circ i_1 \circ h_1^{-1}$ is homotopic to the inclusion map $i_1 \times \mathrm{id}_Q$: $N_1 \times Q \to N_2 \times Q$. Therefore, by Lemma C, there is a homeomorphism

$$\psi_2: N_2 \times Q \to N_2 \times Q$$

such that $\psi_2 \circ h_2' \circ i_1 \circ h_1^{-1} = i_1 \times \mathrm{id}_Q$.

So, if we define $h_2'' = \psi_2 \circ h_2'$, then the following diagram commutes.

$$\begin{array}{ccc} N_1 & \xrightarrow{i_1} & N_2 \\ \downarrow h_1 & & \downarrow h_2'' \\ N_1 \times Q & \xrightarrow{i_1 \times \mathrm{id}} & N_2 \times Q \end{array}$$

However, we may lose the property $h''(X_2) \subset N_2 \times 0$.

Now, we are going to fix h_2'' to obtain a homeomorphism h_2 as we desired.

Consider the Z-set $(N_1 \times Q) \cup h_2''(X_2) = h_2''(N_1 \cup X_2)$ in $N_2 \times Q$. It is clear that $h_2''(X_2 - X_1) = h''(X_2) - (N_1 \times Q)$ and we can homotope $h_2''(X_2)$ into $N_2 \times 0$ (rel. $h_2''(X_1)$). Then, by the relative Z-embedding approximation theorem [**1**, Theorem 18.2], we can assume that the final map of this homotopy is a Z-embedding. Moreover, this homotopy can be extended over $N_1 \times Q$ by the identity.

Again by Lemma C, there is a homeomorphism ϕ_2: $N_2 \times Q \to N_2 \times Q$ such that

(1) $\phi_2 | N_1 \times Q = \mathrm{id}$,

(2) $\phi_2(h_2''(X_2)) \subset N_2 \times 0$.

Thus, if we define $h_2 = \phi_2 \circ h_2''$, then h_2 has the desired properties (a) and (b).

3. *Construction of h_n.* h_n is constructed similarly from h_{n-1} when $n > 3$.

The proof of Proposition 1 is now complete.

LEMMA 1. *Let $\{X_n\}$ be a sequence of Hausdorff spaces such that X_n is a subspace of X_{n+1}, and let $X = \varinjlim X_n$. If K is a compact subspace of X, then there is an integer k such that $K \subset X_k$.*

A closed subset X of M is said to be *strongly negligible* if for each open cover α of M there is a homeomorphism f: $M \to M - X$ such that f is α-close to id_M.

COROLLARY 1. *Every compact subset K of a Q^∞-manifold M is Q^∞-deficient, and therefore strongly negligible.*

PROOF. Consider $M = \varinjlim M_n$ as in Lemma A. Then, $K \subset M_k$ for some k, and K is a Z-set in M_n, $n > k$. It follows that K is an inductive Z-set, and therefore Q^∞-deficient in M. By [5, Theorem 4.2], every Q^∞-deficient set in M is strongly negligible.

PROPOSITION 2. *If X and Y are Q^∞-deficient subsets of a Q^∞-manifold M, then so is $X \cup Y$.*

PROOF. By use of Lemma 1 and Proposition 1 above, we can assume that $M = \varinjlim M_i$ and $M = \varinjlim M_j'$ such that the following hold.

(1) $X_i = X \cap M_i$ is a Z-set in M_i, for each $i = 1, 2, \ldots$.
(2) $Y_j = Y \cap M_j'$ is a Z-set in M_j', for each $j = 1, 2, \ldots$.
(3) M_i is a Z-set of M_i', $i = 1, 2, \ldots$.
(4) M_j' is a Z-set of M_{j+1}, $j = 1, 2, \ldots$.
(5) $M_i - Y$ is not empty for each $i = 1, 2, \ldots$.

We will construct a sequence of compact Q-manifolds M^j such that

(i) $M_j' \subset M^j \subset M_{j+1} \subset M_{j+1}' \subset M^{j+1} \ldots$, for all $j = 1, 2, \ldots$,
(ii) M^j is a Z-set of M^{j+1}, and
(iii) $(X \cup Y) \cap M^j$ is a Z-set in M^j.

Then the proposition follows from Proposition 1, since $X \cup Y$ is now an inductive Z-set in $M = \varinjlim M^j$.

a. *Construction of M^1.* Recall X_1 and Y_1 are Z-sets in M_1'. However, $X_1' = X \cap M_1'$ is not necessarily a Z-set in M_1'.

Since M_1' is a Z-set in M_2, $M_1' - Y$ is a Z-set in $M_2 - Y$ (Lemma B). By the collar theorem [1, Theorem 16.2], there is an open embedding

$$\lambda\colon (M_1' - Y) \times [0, 1) \to M_2 - Y$$

such that $\lambda(x, 0) = x$ for all $x \in M_1' - Y$.

Let $\phi\colon M_1' \to [0, \frac{1}{2}]$ be a map such that $\phi^{-1}(0) = Y_1$. Then

Claim. The pinched collar $M^1 = \{\lambda(x, t) | x \in M_1' - Y, 0 < t < \phi(x)\} \cup Y_1$ of M_1' in M_2 at Y_1 has the following properties.

(1) M^1 is a Q-manifold homeomorphic to M_1', since the natural projection $p\colon M^1 - Y \to M_1' - Y$ may be approximated by a homeomorphism which extends by the identity on Y_1.

(2) $Y_1 = Y \cap M^1$ and Y_1 is a Z-set in M^1, since $Y_1 \subset M_1'$ is a Z-set.

(3) $X^1 = X \cap M^1$ is a Z-set in M^1.

In fact, we can write X^1 as

$$X^1 = [X \cap (M^1 - Y_1)] \cup (X \cap Y_1).$$

Then, $X \cap Y_1$ is a Z-set in M^1. Moreover, we can easily see that $X \cap (M^1 - Y_1)$ is the union of a sequence of compact Z-sets in M^1. Thus, the property (3) follows from Lemma B(i).

b. *Construction of M^j.* Similarly, the Q-manifold M^j will be a pinched collar of M_j' in M_{j+1} at Y_j such that $(X \cup Y) \cap M^j = X^j \cup Y_j$ is a Z-set in M^j. Moreover, we can show that M^j is a Z-set of M^{j+1} as follows. For fixing our idea, we assume that $j = 1$ and we have the following inclusions

$$M^1 \subset M_2 \subset M_2' \subset M^2.$$

Observe that $M_2' - Y = M_2' - Y_2$ is a Z-set in $M^2 - Y = M^2 - Y_2$, since M^2 is a pinched collar of M_2' in M_3 at Y_2. Hence, $M^1 - Y = M^1 - Y_1$ is a Z-set in $M^2 - Y$, since $M^1 - Y$ is closed in $M_2' - Y$. Using Lemma B(ii), we see that $M^1 - Y_1$ is the union of sequences of Z-sets in M^2. Then, Lemma B(i) again shows that $M^1 = (M^1 - Y_1) \cup Y_1$ is a Z-set in M^2, since $Y_1 \subset Y_2$ is also a Z-subset of M^2. Therefore, all desired properties of the sequence $\{M^j\}$ are verified.

We note that in contrast to Lemma B(i), the closed countable union of Q^∞-deficient subsets of a Q^∞-manifold need not be Q^∞-deficient. For example, consider $Q^\infty = \varinjlim Q_n$ and the countable union $\bigcup_1^\infty Q_n = Q^\infty$.

3. Unknotting theorem. Let α be an open cover of M and $U \in \alpha$. We define $\mathrm{St}(U, \alpha) = \bigcup \{V \in \alpha \mid U \cap V \neq \varnothing\}$. By $\mathrm{St}(\alpha)$ we will mean the cover $\{\mathrm{St}(U, \alpha) \mid U \in \alpha\}$. Inductively, we define $\mathrm{St}^{n+1}(\alpha) = \mathrm{St}(\mathrm{St}^n(\alpha), \alpha)$.

MAIN THEOREM. *Let X be a Q^∞-deficient subset of a Q^∞-manifold M and let f: $X \to M$ be a Q^∞-deficient embedding homotopic to the inclusion map i: $X \subset M$. Then, there is an isotopy F: $M \times I \to M$ such that the following hold.*

(i) $F_0 = \mathrm{id}_M$.

(ii) $F_1|X = f$.

(iii) *Given an open cover α of M, if the homotopy H: $i \simeq f$ is limited by α, then the isotopy F can be chosen to be limited by* $\mathrm{St}^4(\alpha)$.

(iv) *Moreover, if $H_t|X_0 =$ inclusion on a closed subset X_0 of X for each $t \in I$, then F_t can be chosen to be the identity on X_0 for each t.*

PROOF. *Case* 1. *We assume that $f(X - X_0) \cap (X - X_0) = \varnothing$.*

Let $\tilde{X}$ denote the quotient space $(X \times I)/\sim$, where $(x, t) \sim (x, 0)$ if $x \in X_0$. Let us define $\tilde{H}$: $\tilde{X} \to M$ by $\tilde{H}([x, t]) = H(x, t)$. By Proposition 2, the restriction of $\tilde{H}$ to $X \times \{0, 1\}/\sim$ is a Q^∞-deficient embedding. By the relative approximation theorem in Q^∞-manifolds [5, Theorem 3.3], we can assume that $\tilde{H}$ is a Q^∞-deficient embedding and the induced homotopy, say H, now is limited by $\mathrm{St}(\alpha)$. We think of M as $M \times Q$, and we assume that $H(X \times I) \subset M \times 0$. Let $M = \varinjlim M_n$ as in Lemma A, and let us define

$$Z_k = H^{-1}(M_k \times Q) = \{(x, t) \in X \times I \mid H(x, t) \in M_k \times Q\},$$

and $X_k = p_X(Z_k)$ where p_X: $X \times I \to X$ is the projection. Now, since $\tilde{Z}_k = Z_k/\sim = \tilde{H}^{-1}(M_k \times Q)$ is compact, it follows that Z_k and X_k are compact, and we may assume that $H(X_k \times I) \subset M_{k+1} \times Q$.

By the last part of this proof, we can assume that there is a sequence of compact Q-manifold N_k's such that for each $k = 1, 2, \ldots,$ (a) $N_k \supset M_k \times Q$, (b) N_k is a Z-set in N_{k+1}, (c) H_t: $X_k \to N_k$ is a Z-embedding for each $t \in I$, and (d) $H[(X - X_k) \times I] \cap N_k = \varnothing$. Therefore, the isotopy of $M = \varinjlim N_k$ that we desired will be constructed inductively on the sequence $\{N_k\}$ by use of the fiber-version of the Z-sets unknotting theorem [6] that can be restated as follows. "Let X be a compact Z-subset in a Q-manifold N, and let H_t: $X \to N$ $(t \in I)$ be a continuous family of Z-embeddings such that $H_0(x) = x$, then H extends to an isotopy $\overline{H}$ of N with $\overline{H}_0 = \mathrm{id}_N$."

Now, to complete the proof, we will construct such a sequence $\{N_k\}$ from the given sequence $\{M_k \times Q\}$ as follows. For fixing our idea, let us work with $k = 1$. Define $Y = (M_1 \times Q) \cup H(X_1 \times I)$. Then, Y is a Z-set in $M_2 \times Q$, since both $M_1 \times Q$ and $H(X_1 \times I)$ are Z-sets in $M_2 \times Q$. Let W be a compact Q-manifold neighborhood of a copy of Y in Q such that (i) Y is a Z-set in W, and (ii) the inclusion map $Y \subset M_2 \times Q$ has an extension over W, say g. Since Y is a Z-set in $M_2 \times Q$, we can assume, by the relative Z-embedding approximation theorem [**1**, Theorem 18.2], that g is a Z-embedding such that $g(W - Y) \cap Y = \varnothing$. If we define $N_1 = g(W)$, and similarly for N_k $(k > 1)$, then the sequence $\{N_k\}$ has all above properties. Hence, the proof of Case 1 is now complete.

Case 2. *General case*. Assuming again that $X \cup f(X) \subset M \times 0$, there is a Q^∞-deficient embedding f': $X \to M \times Q^\infty$ such that

(i) $f'|X_0 =$ inclusion,

(ii) $f'(X - X_0) \cap (M \times 0) = \varnothing$, and

(iii) f' is α-homotopic to f (rel. X_0).

So, f' is $\mathrm{St}(\alpha)$-close to the inclusion $X \subset M \times Q^\infty$.

Using Case 1 twice, we will obtain two isotopies: G^1 carrying the inclusion to f', limited by $\mathrm{St}^2(\alpha)$; and G^2 carrying f' to f, limited by $\mathrm{St}(\alpha)$. Then, the combination of G^1 and G^2 is the isotopy, limited by $\mathrm{St}^4(\alpha)$, that we desired; and the proof is complete.

The proof of the following corollary is similar to that of Lemma 37.1 in [**1**].

COROLLARY 2 (RELATIVE TRIANGULATION THEOREM). *Let (M, M_0) be a pair of Q^∞-manifolds, where M_0 is a Q^∞-deficient subset of M, and let h_0: $M_0 \to K_0 \times Q^\infty$ be a triangulation of M_0. Then, there is a triangulation h: $M \to K \times Q^\infty$ of M such that K_0 is a subcomplex of K and h extends h_0.*

4. Knotting Z-sets in Q^∞ and R^∞. To conclude this paper, we will give a negative answer to the question NLC 6 in [**2**] which asks if there are Z-set unknotting theorems for Q^∞-manifolds and R^∞-manifolds. Before proceeding with the construction, let us set up some notation.

Define $Q_n = Q \times I_1 \times I_2 \times \cdots \times I_n$ and identify it with $Q_n \times \{0\}$ in Q_{n+1}. Let $A_n = Q \times \prod_1^n [0, 1/2^n]_i$ and $\mathring{A}_n = Q \times \prod_1^n [0, 1/2^n)_i$ be subsets of Q_n. Let R^∞ denote the direct limit space $\varinjlim R^n$. By the main result of [**4**], R^∞ is homeomorphic to $\varinjlim \mathring{I}^n$, where $\mathring{I}^n = \prod_1^n [0, 1)_i$. We also define similarly the notion of R^∞-deficiency in R^∞.

LEMMA 2. *The subset $X = \bigcup_1^\infty A_n$ of Q^∞ is a Z-set in Q^∞. However, X is not Q^∞-deficient in Q^∞.*

COROLLARY 3. *The unknotting theorem for Z-sets in Q^∞ is false.*

PROOF OF COROLLARY 3. Recall that every Q^∞-deficient subset of Q^∞ is also a Z-set. Now, we re-embed X into Q^∞ by a Q^∞-deficient embedding. If the unknotting theorem were true, then X would be Q^∞-deficient. This contradicts Lemma 2 above.

PROOF OF LEMMA 2. *Part* 1. *X is a Z-set in Q^∞.*

Fact 1. *$X - (Q \times \{0\})$, where $0 \in \varinjlim I^n$, is a Q^∞-deficient subset of $Q^\infty - (Q \times \{0\})$.*

Set $M_n = Q_n - \mathring{A}_n = Q \times [I^n - \Pi_1^n [0, 1/2^n)]$. Then, M_n has the following properties:

(1) M_n is a compact Q-manifold,

(2) $X \cap M_n = (A_1 \cup A_2 \cup \cdots \cup A_n) - \mathring{A}_n$ is a Z-set in M_n, and

(3) $\varinjlim M_n = \bigcup_1^\infty M_n = Q \times (I^\infty - \{0\}) = Q^\infty - (Q \times \{0\})$.

Therefore, $X - (Q \times \{0\}) = \bigcup_1^\infty (X \cap M_n)$ is an inductive Z-set of $Q^\infty - (Q \times \{0\})$. Thus, it is Q^∞-deficient by Proposition 1.

Fact 2. *$Q \times \{0\}$ is Q^∞-deficient in Q^∞, so it is a Z-set.*

Now, we will complete the proof of Part 1. Given an open cover α of Q^∞, we will construct a map $f: Q^\infty \to Q^\infty - X$ which is $\mathrm{St}(\alpha)$-close to the identity as follows.

First, by Fact 2 there is a map $g: Q^\infty \to Q^\infty - (Q \times \{0\})$ which is α-close to id. Secondly, if α_0 denotes the open cover $\{U \cap (Q^\infty - (Q \times \{0\})) | U \in \alpha\}$ of $Q^\infty - (Q \times \{0\})$, then from Fact 1 there is a map $h: Q^\infty - (Q \times 0) \to Q^\infty - X$ which is α_0-close to id. Finally, the map $f = h \circ g: Q^\infty \to Q^\infty - X$ will be $\mathrm{St}(\alpha)$-close to id.

Part 2. *X is not Q^∞-deficient in Q^∞.*

Assume that $\varinjlim Q^j$ is a decomposition of Q^∞ as in Lemma A. Then by Lemma 1 there is a sequence $\{n_j | j = 1, 2, \ldots\}$ such that $Q^j \subset Q_{n_j}$ for each $j = 1, 2, \ldots$.

Now, it is clear that $X \cap Q_{n_j} \supset \mathring{A}_{n_j}$ and $\mathring{A}_{n_j}$ is open in Q_{n_j}. So, since Q^j is a subspace of Q_{n_j}, the interior of $X \cap Q^j$ in Q^j contains the open set $\mathring{A}_{n_j} \cap Q^j$ of Q^j. Hence, $X \cap Q^j$ is not a Z-set in Q^j when $X \cap Q^j \neq \emptyset$. Since this is true for each decomposition, as in Lemma A, of Q^∞, it follows that X is not an inductive Z-set in Q^∞. Hence, it is not Q^∞-deficient.

The proof of Lemma 2 is complete.

COROLLARY 4. *The unknotting theorem for Z-sets in R^∞ is false.*

PROOF. Think of R^∞ as $\varinjlim \mathring{I}^n$. Let $Y = \bigcup_1^\infty B_n$, where $B_n = \Pi_{i=1}^n [0, 1/2^n]_i$.

Just as in Part 2 of the proof of Lemma 2, we can show that if $R^\infty = \varinjlim M_n$, where M_n is a compact manifold as in Lemma A, then for some n, the interior of $Y \cap M_n$ in M_n is not empty. Thus, Y cannot be R^∞-deficient in R^∞. Moreover, since $Q^\infty - X = Q \times (R^\infty - Y)$ and since X is a Z-set in Q^∞ by Lemma 2, we can show easily that Y is a Z-set in R^∞. The corollary follows by the same argument as for Corollary 3.

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