PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 1, May 1981

A mod p WHITEHEAD THEOREM

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ABSTRACT. A mod p Whitehead theorem is proved which is the relative version of a basic result of localization theory. It is applied to give a family of fibrations which are also cofibrations.

0. Introduction. The classic work of Serre showed how one could generalize the Hurewicz and Whitehead theorems. That is, when $\mathcal{C}$ is a Serre class of abelian groups one obtains both theorems "mod $\mathcal{C}$". Later the work of Sullivan and others on localizations showed that even if $\mathcal{C}$ were not a Serre class the Hurewicz theorem might still hold mod $\mathcal{C}$. Of course we have in mind here letting $\mathcal{C} = \mathcal{C}_l$ the class of all l-local abelian groups; though $\mathcal{C}_l$ is not a Serre class a fundamental result [**7**, p. 18] of localization theory shows the validity of the Hurewicz theorem mod $\mathcal{C}_l$. In this note we establish the relative version of this theorem–that is, we prove [Theorem 3.1 below] a Whitehead theorem mod $\mathcal{C}_l$. We shall in fact work in the more general setting of nilpotent spaces and groups. Our proof uses in a natural way the technique of p-completions.

We finish by giving a curious application of our result. In §4 we consider the question raised by Milgram [**6**, p. 246] "what fibrations are also cofibrations?"

1. Algebraic preliminaries. By group we always mean nilpotent group although we write the group operation additively. We first recall some terminology. Let G be a group and p be a prime. Say that G is p-divisible (resp. has no p-torsion) (resp. is uniquely p-divisible) if the function "multiplication by p" $\times p : G \to G$ is onto (resp. one-to-one) (resp. bijective). If $l \subset P$ where P denotes the set of all primes then G is said to be l-local if G is uniquely p-divisible for each $p \in P - l$. By the p-completion of G we always mean the ext p-completion $\operatorname{Ext}(Z_{p^\infty}, G)$ which we write as $\overline{G}_p$. See [**1**, Chapter VI] for facts about p-completion. A group G is said to be p-complete if the completion homomorphism $G \to \overline{G}_p$ is bijective and if $\operatorname{Hom}(Z_{p^\infty}, G) = 0$ [**1**, p. 172, 3.1]. We point out that the condition $\operatorname{Hom}(Z_{p^\infty}, G) = 0$ is redundant since by [**1**, p. 172, 3.2] $\overline{G}_p$ is p-complete; hence $\operatorname{Hom}(Z_{p^\infty}, G) \simeq \operatorname{Hom}(Z_{p^\infty}, \overline{G}_p) = 0$.

We will make use of the following four lemmas, the first due to Bousfield and Kan [**1**, p. 176, 3.6].

LEMMA 1.1. *A group G is p-divisible if and only if $\overline{G}_p = 0$.*

Received by the editors September 5, 1979 and, in revised form, August 11, 1980.
1980 *Mathematics Subject Classification*. Primary 55D99, 18G18; Secondary 55D05, 55F05.

LEMMA 1.2. *If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{j} E$ is an exact sequence of groups with* $\operatorname{im} j$ *a normal subgroup of E and where A, B, D, and E are p-complete, then so is C p-complete.*

PROOF. We outline a sketch only. One first establishes that the lemma holds for short exact sequences, i.e., with $A = E = 0$. One can also show that whenever $k: G \to H$ is a homomorphism with normal image between two p-complete groups G and H then so are the groups ker k, im k, and coker k again p-complete. The lemma is then proved by taking the exact sequence in its statement and passing to the associated short exact sequence $0 \to \operatorname{coker} f \to C \to \ker j \to 0$.

LEMMA 1.3. *A group G is uniquely p-divisible if and only if both $\bar{G}_p = 0$ and* $\operatorname{Hom}(Z_{p^\infty}, G) = 0$.

PROOF. The implication $\Rightarrow$ follows from Lemma 1.1. To prove the converse we shall show that G is isomorphic to $\operatorname{Hom}(Z[1/p], G)$ which is always uniquely p-divisible. The exact sequence of abelian groups $0 \to Z \xrightarrow{i} Z[1/p] \to Z_{p^\infty} \to 0$ gives rise to the exact sequence of groups

$$\operatorname{Hom}(Z_{p^\infty}, G) \to \operatorname{Hom}\left(Z\left[\frac{1}{p}\right], G\right) \xrightarrow{i^*} \operatorname{Hom}(Z, G) \simeq G.$$

By [1, p. 177, 3.7] the subgroup $\operatorname{im}(i^*)$ is the kernel of the completion homomorphism $G \to \bar{G}_p$; thus $\bar{G}_p = 0$ implies i^* is onto. And $\operatorname{Hom}(Z_{p^\infty}, G) = 0$ implies i^* is one-to-one.

The final lemma will be used in computing obstructions:

LEMMA 1.4. *If A and G are abelian groups with G p-complete then*
(a) $\operatorname{Hom}(A, G) = 0$ *if A is p-divisible.*
(b) $\operatorname{Ext}(A, G) = 0$ *if A has no p-torsion.*

PROOF. (a) Assume to the contrary that $f: A \to G$ is a nontrivial homomorphism. Since A is p-divisible so is $\operatorname{im} f$ again p-divisible. We could then construct a monomorphism $g: Z[1/p] \to \operatorname{im} f$, and hence a nontrivial homomorphism from $Z[1/p]$ into G. But this contradicts the fact that $\operatorname{Hom}(Z[1/p], G) = 0$ by [1, p. 174, 3.4].

(b) $\operatorname{Ext}(A, G) \simeq \operatorname{Ext}(A, \operatorname{Ext}(Z_{p^\infty}, G)) \simeq \operatorname{Ext}(\operatorname{Tor}(A, Z_{p^\infty}), G) \simeq \operatorname{Ext}(0, G) = 0$. (The second isomorphism from the left is a general identity [2, p. 116].)

2. Topological preliminaries. First we set up some terminology. By space we always mean a nilpotent space of the homotopy type of a connected CW complex. By the p-completion of a space X we always mean the ext p-completion space $\bar{X}_p$ in the sense of Bousfield and Kan [1, Chapter VI]. We have the associated p-completion map $\delta: X \to \bar{X}_p$, and a space X is called p-complete if δ is a homotopy equivalence. Recall that X is p-complete if and only if all its homotopy groups are p-complete groups [1, p. 184, 5.4].

It is convenient to follow Hilton [3] in his use of the homology and homotopy groups of a map $f: X \to Y$. For example, if M_f and E_f denote the mapping cylinder and mapping track (respectively) of f then the groups $H_n(f) \simeq H_n(M_f, X)$ and $\pi_n(f) \simeq \pi_n(Y, E_f)$. In general $\pi_1(f)$ is not a group; however we will require in §3 that f induce a surjection of fundamental groups which insures that $\pi_1(f) = 0$.

We shall need the following two results, the first being due to Bousfield and Kan [**1**, p. 183].

LEMMA 2.1. *If X is a space then the following is a split short exact sequence of groups*:

$$0 \to \overline{\pi_n(X)_p} \to \pi_n(\bar{X}_p) \to \mathrm{Hom}(Z_{p^\infty}, \pi_{n-1}(X)) \to 0.$$

PROPOSITION 2.2. *Let $g: X \to Y$ be a map of p-complete spaces. Then, for $N > 2$, the groups $H_r(g; Z/pZ) = 0$ for all $r \leqslant N$ if and only if the groups $H_r(g) = 0$ for all $r \leqslant N$.*

PROOF. $\Leftarrow$Universal Coefficient Theorem for homology.

$\Rightarrow$ In preparation for this proof we present two lemmas:

LEMMA 2.3. *The implication $\Rightarrow$ is true if in addition to the hypotheses above X and Y are 1-connected.*

PROOF OF LEMMA 2.3. Assume by induction that for some n, $1 \leqslant n < N$, the lemma holds for $r \leqslant n$. Since X and Y are p-complete so are their homotopy groups. Then $\pi_{n+1}(g)$ is a p-complete group by Lemma 1.2 so that by the inductive hypothesis and the Relative Hurewicz Theorem for 1-connected spaces $H_{n+1}(g) \simeq \pi_{n+1}(g)$ is a p-complete group. On the other hand since $H_{n+1}(g; Z/pZ) = 0$ the Universal Coefficient Theorem for homology gives that $H_{n+1}(g)$ is a p-divisible group. But any p-complete, p-divisible group is trivial by Lemma 1.1; therefore $H_{n+1}(g) = 0$ and the inductive step is complete.

In order to drop the 1-connectedness assumption we must prove the following technical lemma.

LEMMA 2.4. *Let (α, β) be a pair of maps $\alpha: X \to Y$, $\beta: X \to Z$ such that $H_r(\alpha; Z/pZ) = 0$ for all $r \leqslant N$, and Z is a p-complete space with $\pi_r(Z) = 0$ for all $r \geqslant N$. Then β extends to a map $\gamma: Y \to Z$ so that $\gamma \circ \alpha = \beta$.*

PROOF OF LEMMA 2.4. We use obstruction theory. Since Z is nilpotent it has a "long" Postnikov decomposition as a tower of principal fibrations $q_s^r: Z_s^r \to Z_{s-1}^r$. Each q_s^r is induced by a map $Z_{s-1}^r \to K(A_s^r, r+1)$ where A_s^r is an abelian quotient in any given central series for $\pi_r(Z)$ under the nilpotent operation of $\pi_1(Z)$. All obstructions to extending β to γ lie in groups of the form $H^{r+1}(\alpha; A_s^r)$.

The Universal Coefficient Theorem for cohomology gives $H^{r+1}(\alpha; A_s^r) \simeq \mathrm{Ext}(H_r(\alpha), A_s^r) \oplus \mathrm{Hom}(H_{r+1}(\alpha), A_s^r)$. By our hypotheses and the Universal Coefficient Theorem for homology the groups $H_r(\alpha)$ are p-divisible for $r \leqslant N$, and have no p-torsion for $r < N$. Also since Z is p-complete it can be shown that all A_s^r in the *upper* central series are p-complete (abelian) groups (cf. [**1**, p. 174, 3.4(ii)] in the case $r = 1$). We may therefore use Lemma 1.4 to conclude $H^{r+1}(\alpha; A_s^r) = 0$ for all $r < N$. And if $r \geqslant N$ then $\pi_r(Z) = 0$ so that each $A_s^r = 0$–i.e. $H^{r+1}(\alpha; A_s^r) = 0$. Hence all obstructions vanish and β extends to γ as claimed.

Proof of Proposition 2.2. Let $\beta_w: W \to W^{[n-1]}$ denote the n-equivalence of a space W into its $[n-1]$-Postnikov section $W^{[n-1]}$. If W is p-complete so is $W^{[n-1]}$. We apply Lemma 2.4 to the pair (g, β_x) to obtain a map $\gamma: Y \to X^{[N-1]}$ so that $\gamma \circ g = \beta_x$. It follows that $H_r(\gamma; Z/pZ) = 0$ for $r \leqslant N$, and that γ is surjective on homotopy groups in all dimensions. Next reapply Lemma 2.4 to the pair (γ, β_y) to obtain a map $h: X^{[N-1]} \to Y^{[N-1]}$ with $h \circ \gamma = \beta_y$. Then γ is seen to be injective on homotopy groups in dimensions r, $r \leqslant N-1$; hence h is in fact a homotopy equivalence. Note that at this point we could conclude that $H_r(g) = 0$ for all $r < N$ which is almost our goal. To show $H_N(g) = 0$ we must work a bit harder.

To this end consider the ladder of nilpotent fibrations

$$\begin{array}{ccccc} F_x & \to & X & \to & X^{[N-1]} \\ \downarrow f & & \downarrow g & & \downarrow h \\ F_y & \to & Y & \to & Y^{[N-1]} \end{array}$$

where F_x and F_y are the fibres of β_x and β_y respectively and f is the induced map of fibres. The methods of the Hilton-Roitberg generalization of the Spectral Sequence Comparison Theorem to nilpotent fibrations [**4**] may be applied to study the homology with Z/pZ coefficients of this ladder. We observe that the absolute (trivial Serre class) version of Theorem 3.2 of [**4**, p. 440] is valid here in the case of homology with Z/pZ coefficients. In particular since h is a homotopy equivalence and $H_r(g; Z/pZ) = 0$ for $r \leqslant N$, we conclude that $H_r(f; Z/pZ) = 0$ for $r \leqslant N$.

Since F_x and F_y are again p-complete spaces by Lemma 1.2, and are at least 1-connected, we may apply Lemma 2.3 to get that $H_r(f) = 0$ for $r \leqslant N$. Finally, again applying the Comparison Theorem, $H_r(g) = 0$ for $r \leqslant N$. Q.E.D.

3. Proof of the main result. This is the mod p Whitehead theorem. Recall that all spaces are nilpotent.

Theorem 3.1. *Let $f: X \to Y$ be a map of spaces which induces a surjection of fundamental groups. The following two conditions are equivalent for $N \geqslant 2$:*

(1) $H_r(f)$ is a p-divisible group for each $r \leqslant N$ and has no p-torsion for each $r < N$.

(2) $\pi_r(f)$ is a p-divisible group for each $r \leqslant N$ and has no p-torsion for each $r < N$.

Proof. Let F denote the fibre of $f: X \to Y$ and note that F is connected and nilpotent. Consider the ladder of fibrations [**1**, p. 187, 6.5(i)]

$$\begin{array}{ccccc} F & \to & X & \xrightarrow{f} & Y \\ \delta_1\downarrow & & \delta_2\downarrow & & \delta_3\downarrow \\ \bar{F}_p & \to & \bar{X}_p & \xrightarrow[g]{} & \bar{Y}_p \end{array}$$

where δ_i denotes the appropriate p-completion for $1 \leqslant i \leqslant 3$. We now list a series of equivalent statements. The parenthetical comment after each statement i is to justify the equivalence of statements i and $i-1$.

1. Condition (1) of the theorem.

2. $H_r(f; Z/pZ) = 0$ for each $r \leqslant N$. [Universal Coefficient Theorem for homology].

3. $H_r(g; Z/pZ) = 0$ for each $r \leqslant N$. [The maps δ_2 and δ_3 always induce isomorphisms on homology with Z/pZ coefficients [**1**, p. 186, 6.1]].

4. $H_r(g) = 0$ for each $r \leqslant N$. [Proposition 2.2].

5. $\pi_r(g) = 0$ for each $r \leqslant N$. [Relative Hurewicz Theorem for nilpotent spaces [**4**, p. 441, Corollary 3.4]].

6. $\pi_r(\bar{F}_p) = 0$ for each $r \leqslant N - 1$. [The group $\pi_r(\bar{F}_p) \simeq \pi_{r+1}(g)$].

7. $\overline{\pi_r(F)_p} = 0$ for each $r \leqslant N - 1$ and $\mathrm{Hom}(Z_{p^\infty}, \pi_r(F)) = 0$ for each $r < N - 1$. [Lemma 2.1].

8. $\pi_r(F)$ is p-divisible for each $r \leqslant N - 1$ and has no p-torsion for each $r < N - 1$. [Lemmas 1.1 and 1.3].

9. Condition (2) of the theorem. [The group $\pi_r(F) \simeq \pi_{r+1}(f)$]. Q.E.D.

We remark that if the spaces X and Y of Theorem 3.1 are 1-connected of finite type, then the theorem is known as a result of $\mathcal{C}$-theory. For in this case condition (1) (resp. condition (2)) of the theorem is equivalent to the condition that the group $H_r(f)$ (resp. $\pi_r(f)$) is contained in the Serre class of torsion abelian groups having no element with order a power of p for each $r \leqslant N$.

However one is also interested in this theorem for spaces not of finite type. For we may then interpret 3.1 as a relative version of the localization theory result that states that a space X has l-local homology groups if and only if X has l-local homotopy groups [**7**, p. 18], [**5**, p. 72, Theorem 3B]. To see this fact let $p \in P - l$ and set $Y = *$ in 3.1.

4. Fibrations which are also cofibrations. Milgram [**6**. p. 246] pointed out the "mysterious result" that the fibration $K(Q/Z, n) \to K(Z, n + 1) \to K(Q, n + 1)$ is also a cofibration. Theorem 3.1 leads to the following generalization of his example (which is then recovered by taking $X = K(Z, n + 1)$ and $l = \varnothing$).

THEOREM 4.1. *Let $j\colon X \to X_l$ be the localization of a 1-connected space X at a set of primes l, and let $i\colon F \to X$ denote inclusion of the (homotopy theoretic) fibre F of the map j into X. Then the fibration $F \xrightarrow{i} X \xrightarrow{j} X_l$ is also a cofibration.*

PROOF. Let $\rho\colon X \to C$ denote the projection of X onto the cofibre C of the map $i\colon F \to X$. Then there exists a map $h\colon C \to X_l$ to make the following a homotopy commutative diagram:

$$\begin{array}{ccccc} F & \xrightarrow{i} & X & \xrightarrow{\rho} & C \\ \downarrow 1 & & \downarrow 1 & & \downarrow h \\ F & \xrightarrow{i} & X & \xrightarrow{j} & X_l \end{array}$$

We shall show, to prove the theorem, that the map h is a homotopy equivalence.

To this end apply the localization functor to the diagram above to obtain another homotopy commutative diagram:

$$
\begin{tikzcd}
F \arrow[rr, "i"] \arrow[dr] \arrow[dd] & & X \arrow[rr, "\rho"] \arrow[dr, "j"] \arrow[dd, "1" near start] & & C \arrow[dr, "j"] \arrow[dd, "h" near start] & \\
& F_l \arrow[rr, crossing over] & & X_l \arrow[rr, "\rho_l", crossing over] & & C_l \arrow[dd, "h_l"] \\
F \arrow[rr] \arrow[dr] & & X \arrow[rr, "j" near end] \arrow[dr, "j"'] & & X_l \arrow[dr, "j"'] & \\
& F_l \arrow[rr] \arrow[from=uu, crossing over] & & X_l \arrow[rr, "j_l"'] \arrow[from=uu, "1" near end, crossing over] & & X_l
\end{tikzcd}
\tag{4.2}
$$

In this diagram j denotes the appropriate localization maps, and if $f\colon X \to Y$ is a map then $f_l\colon X_l \to Y_l$ denotes the map induced by f upon localization.

We first study the map $j\colon C \to C_l$. Since $F \xrightarrow{i} X \xrightarrow{j} X_l$ is a fibration, $\pi_*(i) \simeq \pi_*(X_l)$. But $\pi_*(X_l)$ is l-local and the map i induces a surjection of fundamental groups; therefore by Theorem 3.1 the group $H_*(i)$ is again l-local. And since $F \xrightarrow{i} X \xrightarrow{\rho} C$ is a cofibration, $H_*(i) \simeq H_*(C)$ so that C is an l-local space. Thus $j\colon C \to C_j$ is a homotopy equivalence.

We next establish that the map $h_l\colon C_l \to X_l$ is a homotopy equivalence. In this regard, recall that the localization functor preserves both fibrations and cofibrations so that $F_l \to X_l \xrightarrow{j_l} X_l$ is a fibration and $F_l \to X_l \xrightarrow{\rho_l} C_l$ is a cofibration. But since $j_l\colon X_l \to X_l$ is easily seen to be a homotopy equivalence, the space $F_l \equiv *$. Thus the map $\rho_l\colon X_l \to C_l$ is a homotopy equivalence. Then $h_l\colon C_l \to X_l$ is a homotopy equivalence by considering diagram (4.2).

Again referring to diagram (4.2) we see that since $j\colon C \to C_l$, $h_l\colon C_l \to X_l$, and $j\colon X_l \to X_l$ are homotopy equivalences, so is $h\colon C \to X_l$ also a homotopy equivalence.

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