PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 1, May 1981

MORE COUNTEREXAMPLES TO COLEMAN'S CONJECTURE

DENNIS PIXTON[1]

ABSTRACT. For any $m, n \geqslant 2$ we construct a smooth vector field with a topologically hyperbolic equilibrium of type (m, n) which is not locally topologically conjugate to a linear vector field. This refutes Coleman's conjecture in all cases not covered by previous work of Neumann, Walker, and Wilson.

Courtney Coleman's conjecture **[C]** is that a suitable topological notion of hyperbolicity for an equilibrium point x_0 of a vector field X should guarantee that X is topologically conjugate, near x_0, to a linear hyperbolic vector field. Wes Wilson **[Wi1]** clarified this conjecture by defining topologically hyperbolic equilibria of type (m, n); this definition is reviewed below. He then showed that the reformulated conjecture is true if $m \leqslant 1$ or $n \leqslant 1$. However, Dean Neumann **[N]** produced a counterexample of type (2, 2), and this was extended by Russell Walker **[Wa]** to cover types $(m, 2)$ and $(2, n)$ for $m, n \geqslant 2$. For a survey of these developments see **[Wi2]**.

We prove in this note that the conjecture is false for all types (m, n) with $m, n \geqslant 2$.

First, Wilson's reformulation: Consider the *standard example* X_{mn} given by $X_{mn}(x, y) = (-x, y)$ for $(x, y) \in \mathbf{R}^m \times \mathbf{R}^n$. With respect to $B := D^m \times D^n$, writing $X = X_{mn}$, $\Phi =$ the flow of X, we have

$$\begin{aligned} &\text{(a)}\quad X \text{ points inward on } b^+ := \partial D^m \times D^n,\\ &\text{(b)}\quad X \text{ points outward on } b^- := D^m \times \partial D^n,\\ &\text{(c)}\quad A^+ := D^m \times 0 = \{(x, y) : \Phi_t(x, y) \in B \text{ for all } t \geqslant 0\},\\ &\text{(d)}\quad A^- := 0 \times D^n = \{(x, y) : \Phi_t(x, y) \in B \text{ for all } t \leqslant 0\}. \end{aligned} \tag{1}$$

This is exactly the information one derives from an isolating block analysis of X_{mn}. Recall that two vector fields are topologically conjugate iff some homeomorphism between their domains maps the oriented trajectories of one onto those of the other. We say an equilibrium x_0' of a vector field X' is *topologically hyperbolic of type* (m, n) iff X' restricted to some neighborhood of x_0' is topologically conjugate to some vector field X defined on B and satisfying (1). Well-known arguments show that a linear vector field $x \to Ax$ has a topologically hyperbolic equilibrium at 0 of type (m, n) if and only if it is topologically conjugate to X_{mn}. Hence Coleman's conjecture becomes: Any vector field X satisfying (1) is topologically conjugate to X_{mn}.

Received by the editors January 1, 1980 and, in revised form, July 21, 1980.
AMS (MOS) subject classifications (1970). Primary 58F10, 34C05.
Key words and phrases. Hyperbolic equilibrium, topological hyperbolicity, topological conjugacy.
[1]Research partially supported by NSF Grants MCS77-02509 and MCS79-02765.

0002-9939/81/0000-0228/$02.00

Our result is

THEOREM. *For any $m, n \geqslant 2$ there is a C^∞ vector field X satisfying* (1) *which is not topologically conjugate to X_{mn}.*

Before describing the counterexample we fix some notation. Write $a^\pm = b^\pm \cap A^\pm$. Then $b^+ \setminus a^+ = \partial D^m \times (D^n \setminus 0)$ can be identified with $T := S^{m-1} \times S^{n-1} \times (0, 1]$ via the diffeomorphism $(\xi, \eta, r) \to (\xi, r\eta)$. Similarly $b^- \setminus a^- = (D^m \setminus 0) \times \partial D^n$ can be identified with T using $(\xi, \eta, r) \to (r\xi, \eta)$. When convenient we will regard S^{m-1} as $\mathbf{R}^{m-1} \cup \infty$ and S^{n-1} as $\mathbf{R}^{n-1} \cup \infty$. To signal this coordinatization we shall write (x, y) instead of $(\xi, \eta) \in S^{m-1} \times S^{n-1}$. Our example X will define a Poincaré map ϕ: $b^+ \setminus a^+ \to b^- \setminus a^-$ by following trajectories; ϕ is a diffeomorphism. We write ψ for the Poincaré map of the standard example. A simple calculation shows that ψ: $T \to T$ is the identity.

Now we need a C^∞ diffeomorphism $\bar{\phi}$ of $S^{m-1} \times S^{n-1}$, smoothly isotopic to the identity, with

$$\bar{\phi}(x, \infty) = (x, 0) \quad \text{for } x \in D^{m-1}, \tag{2}$$

$$\bar{\phi}(0, y) = (\infty, y) \quad \text{for } y \in D^{n-1}. \tag{3}$$

To find such $\bar{\phi}$ notice that the inclusion of $(D^{m-1} \times \infty) \cup (0 \times D^{n-1})$ into $S^{m-1} \times S^{n-1}$ is isotopic to the embedding given above (using $m-1, n-1 \geqslant 1$) and apply the Isotopy Extension Theorem [H]. Given Φ, Wilson [Wi2] has a procedure to construct a C^∞ vector field X satisfying (1) with Poincaré map

$$\phi(x, y, r) = \left(\bar{\phi}(x, y), r\right) \quad \text{for } r < 1/2.$$

This is our counterexample.

To derive a contradiction we suppose X is conjugate to X_{mn} and let h: $B \to B$ be a homeomorphism carrying trajectories of X to trajectories of X_{mn}. Set $h^\pm = h|b^\pm$, so $h^-\phi = \psi h^+$. Since h^+ is a homeomorphism of b^+ onto itself preserving $a^+ = S^{m-1} \times 0$ we can define a homeomorphism $\tilde{h}^+$ of a^+ by $(\tilde{h}^+(\xi), 0) = h^+(\xi, 0)$. Similarly we have $\tilde{h}^-$: $S^{n-1} \to S^{n-1}$.

For maps f into T we write $f = (f_1, f_2, f_3)$ corresponding to the factors of T. The following is from [Wa].

LEMMA. *Given $\varepsilon > 0$ there is $r_\varepsilon > 0$ such that, for all $(\xi, \eta, r) \in T$ with $r < r_\varepsilon$,*

$$d\left(h_1^-\phi(\xi, \eta, r), \tilde{h}^+(\xi)\right) < \varepsilon, \tag{4}$$

$$d\left(h_2^-\phi(\xi, \eta, r), \tilde{h}^-\phi_2(\xi, \eta, r)\right) < \varepsilon, \tag{5}$$

$$0 < h_3^-\phi(\xi, \eta, r) < \varepsilon. \tag{6}$$

PROOF. On $b^- = D^m \times S^{n-1}$ we use the metric

$$d((z_1, \eta_1), (z_2, \eta_2)) = d(z_1, z_2) + d(\eta_1, \eta_2)$$

where the distances on the right are Euclidean. By uniform continuity of h^- on b^- there is $r_\varepsilon > 0$ so that, for $\xi \in S^{m-1}$,

$$d(h^-(r\xi, \eta), h^-(0, \eta)) < \varepsilon \quad \text{whenever } r = d((r\xi, \eta), (0, \eta)) < r_\varepsilon.$$

Now, under the identification between T and $b^- \setminus a^-$, and suppressing some notation, $h^-(r\xi, \eta) = (h_3^- h_1^-, h_2^-)$. Recalling $h^-(0, \eta) = (0, \tilde{h}^-(\eta))$, the above becomes

$$\varepsilon > d\big((h_3^- h_1^-, h_2^-), (0, \tilde{h}^-)\big) = d(h_3^- h_1^-, 0) + d\big(h_2^-, \tilde{h}^-\big),$$

so $h_3^-(\xi, \eta, r) < \varepsilon$ and $d(h_2^-(\xi, \eta, r), \tilde{h}^-(\eta)) < \varepsilon$. Since ϕ preserves the r-coordinate these imply (5) and (6). The analogous argument for h^+ gives

$$d\big(h_1^+(\xi, \eta, r), \tilde{h}^+(\xi)\big) < \varepsilon \quad \text{for } r < r_\varepsilon.$$

Recalling $h^- \phi = \psi h^+$ and $\psi =$ identity, we have (4). $\square$

Now define the homeomorphism $G\colon T \to T$ by

$$G = \big((\tilde{h}^+)^{-1} h_1^-, (\tilde{h}^-)^{-1} h_2^-, h_3^-\big).$$

Using the uniform continuity of $(\tilde{h}^\pm)^{-1}$ we can shrink r_ε and transform (4)–(6) into

$$d\big(G_1\phi(\xi, \eta, r), \xi\big) < \varepsilon, \tag{7}$$

$$d\big(G_2\phi(\xi, \eta, r), \phi_2(\xi, \eta, r)\big) < \varepsilon, \tag{8}$$

$$0 < G_3\phi(\xi, \eta, r) < \varepsilon \tag{9}$$

for $r < r_\varepsilon < 1/2$. We will determine ε in retrospect.

Next we require r_1, r_2, r_3, R_1, R_3 with $0 < r_1 < r_2 < r_3 < r_\varepsilon$ and

$$G_3\phi(\xi, \eta, r_1) < R_1 < G_3\phi(\xi', \eta', r_2) < R_3 < G_3\phi(\xi'', \eta'', r_3) \tag{10}$$

for all $\xi, \xi', \xi'' \in S^{m-1}$, $\eta, \eta', \eta'' \in S^{n-1}$. To produce these we first choose $r_3 < r_\varepsilon$; then choose $R_3 < \min G_3\phi(\xi'', \eta'', r_3)$ (which is positive by (9)); then use $\varepsilon = R_3$ in (9) to obtain r_2. Now repeat: choose R_1; then use (9) with $\varepsilon = R_1$ to obtain r_1.

Consider now the two embeddings α^0, α^1 of D^{m-1} into T given by

$$\alpha^0(x) = (x, 0, R_2) \quad \text{where } R_2 = \tfrac{1}{2}(R_1 + R_3),$$

$$\alpha^1(x) = G\phi(x, \infty, r_2).$$

Then $\alpha^s = (1 - s)\alpha^0 + s\alpha$, $0 \leqslant s \leqslant 1$, is a homotopy between α^0 and α^1, and (2), (7), (8), (10) imply

$$\alpha^s(D^{m-1}) \subset N_\varepsilon(D^{m-1}) \times N_\varepsilon(0) \times (R_1, R_3), \tag{11}$$

$$\alpha^s(\partial D^{m-1}) \subset N_\varepsilon(\partial D^{m-1}) \times N_\varepsilon(0) \times (R_1, R_3), \tag{12}$$

for all s, where N_ε means ε-neighborhood. Similarly we define $\beta^0, \beta^1 : E = D^{n-1} \times [r_1, r_3] \to T$ by

$$\beta^0(y, r) = \left(0, y, R_1 + (R_3 - R_1)\frac{r - r_1}{r_3 - r_1}\right),$$

$$\beta^1(y, r) = G\phi(0, y, r).$$

Again we set $\beta^t = (1 - t)\beta^0 + t\beta^1$ and we find, from (3), (7), (8), (10) that, for all $t \in [0, 1]$,

$$\beta^t(E) \subset N_\varepsilon(0) \times N_\varepsilon(D^{n-1}) \times (0, 1], \tag{13}$$

$$\begin{aligned} \beta^t(\partial E) &= \beta^t\big(\partial D^{n-1} \times [r_1, r_3]\big) \cup \beta^t\big(D^{n-1} \times \{r_1, r_3\}\big) \\ &\subset N_\varepsilon(0) \times N_\varepsilon(\partial D^{n-1}) \times (0, 1] \cup \{(\xi, \eta, r) : r \leqslant R_1 \text{ or } r \geqslant R_3\}. \end{aligned} \tag{14}$$

From (11)–(14) we see that, for ε small enough and all $s, t \in [0, 1]$,

$$\alpha^s(D^{m-1}) \cap \beta^t(\partial E) = \alpha^s(\partial D^{m-1}) \cap \beta^t(E) = \varnothing. \tag{15}$$

Finally, $\alpha^0(D^{m-1})$ and $\beta^0(E)$ are transverse and meet only at $(0, 0, R_2)$. Hence, using (15) and the homotopy invariance of intersection number (see [H]), we have $\alpha^1(D^{m-1}) \cap \beta^1(E) \neq \varnothing$. Thus

$$\begin{aligned} \varnothing &\neq G^{-1}\alpha^1(D^{m-1}) \cap G^{-1}\beta^1(E) \\ &= \bar{\phi}(D^{m-1} \times \infty) \times r_2 \cap \bar{\phi}(0 \times D^{n-1}) \times [r_1, r_3], \end{aligned}$$

which contradicts (2), (3). So we are finished.

References

[C] C. Coleman, *Hyperbolic stationary points*, Reports of the First Internat. Congr. Nonlinear Oscillations (Kiev, 1969), Vol. 2, Qualitative Methods in the Theory of Nonlinear Oscillations (Ju. A. Mitropol'skiĭ and A. N. Sarkovskiĭ, eds.), Izdanie Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev, 1970, pp. 222–227.

[H] M. Hirsch, *Differential topology*, Graduate Texts in Math. No. 33, Springer-Verlag, New York, 1976.

[N] D. Neumann, *Topologically hyperbolic equilibria in dynamical systems*, J. Differential Equations **37** (1980), 49–59.

[Wa] R. Walker, *Conjugacies of topologically hyperbolic fixed points: a necessary condition on foliations*, Global Theory of Dynamical Systems (Proc. Northwestern, 1979), edited by Z. Nitecki and C. Robinson, Lecture Notes in Math., vol. 819, Springer, New York, 1980, pp. 446–457.

[Wi1] F. W. Wilson, *A uniform continuity condition which is equivalent to Coleman's Conjecture*, J. Differential Equations **36** (1980), 12–19.

[Wi2] ______, *Coleman's Conjecture on topological hyperbolicity*, Global Theory of Dynamical Systems (Proc. Northwestern, 1979), edited by Z. Nitecki and C. Robinson, Lecture Notes in Math., vol. 819, Springer, New York, 1980, pp. 458–470.

Department of Mathematical Sciences, State University of New York, Binghamton, New York 13901