UNIBRANCHED PRIME IDEALS AND GOING DOWN IN PI RINGS

PHILLIP LESTMANN

ABSTRACT. The purpose of this paper is to answer the question of whether going down is equivalent to unbranchedness of prime ideals in integral extensions of prime PI rings. We show by example that, in general, the answer is no; and we find an additional condition which, together with going down, implies prime ideals of $\text{ht} > 1$ are unbranched.

In [4] McAdam proved the following theorem: *Let R be a commutative Noetherian domain and let T be a domain between R and its integral closure. Then $R \subset T$ has going down if and only if every prime P of R with rank $(\text{ht}) P > 1$ is unbranched in T .*

In this paper we regard a Noetherian prime PI ring as being the (possibly) noncommutative analogue to a commutative Noetherian domain; its central integral closure as defined in [7] as the analogue of integral closure; and ask whether the conclusion of the theorem holds in this case. We show that an additional condition to going down is necessary to imply unbranchedness in the noncommutative case and prove a generalization of the above theorem using this condition. Two counterexamples are given which show that the conditions are also necessary.

By a PI ring we understand an associative ring which satisfies a polynomial identity. Standard results on PI rings may be found in [6]. The usual definitions for lying over (LO), going up (GU), incomparability (INC), and going down (GD) are assumed here. They may be found in [3].

We list below some key definitions.

DEFINITION 1. For any two rings $R \subset S$, let $S^R = \{s \in S \mid sr = rs \text{ for all } r \in R\}$. We say that $R \subset S$ is an *extension* if $S = RS^R$. The extension is *central* if $S = RZ(S)$ where $Z(S) = \{s \in S \mid ts = st \text{ for all } t \in S\}$ (the *center* of S).

DEFINITION 2. Let R be a prime PI ring, Q its ring of quotients, and $K = Z(Q)$. Then the *central integral closure* of R is obtained by adjoining to R all elements of K (= the quotient field of $Z(R)$) which satisfy a monic polynomial with coefficients in R .

DEFINITION 3. Let R be a prime ring; i.e., (0) is a prime ideal. A prime ideal P of R is said to have *height 1* ($\text{ht } 1$) if $P_1 \subset P$, P_1 a prime ideal of R , implies $P_1 = (0)$ or $P_1 = P$. If P properly contains a $\text{ht } 1$ prime, we will say P has $\text{ht} > 1$.

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DEFINITION 4. Let $R \subset S$ be an extension having LO.

(i) If a prime ideal P of R is such that for prime ideals Q and Q_1 of S , $Q \cap R = P$ and $Q_1 \cap R = P$ forces $Q = Q_1$, P is said to be *unbranched* in S .

(ii) If P is a prime ideal of R so that whenever primes Q and Q_1 of S have $Q \cap R = P = Q_1 \cap R$ and $Q \cap Z(S) = Q_1 \cap Z(S)$ it must be that $Q = Q_1$, then P will be called *weakly unbranched* in S .

Observe that unbranched always implies weakly unbranched. In the case of commutative rings all prime ideals are weakly unbranched. Example 1 shows that weakly unbranched does not imply unbranched in general. Example 2, given after the Theorem below, exhibits a central integral extension having GD in which the small ring has a prime ideal that is not weakly unbranched.

EXAMPLE 1. Let $J =$ the integers, $S = J[i] = \{a + bi | i^2 = -1; a, b \in J\}$, and $R = J + 5iJ = \{a + 5bi | a, b \in J\}$. Then $5J + 5iJ$ is a prime ideal of R having two primes of S lying over it. The two primes are $(2 + i)S$ and $(2 - i)S$. Note that S is the integral closure of R in its quotient field. Both rings are Noetherian.

LEMMA. Let R be a Noetherian prime PI ring and Q_1 and Q_2 two prime ideals of R of $\text{ht} > 1$ for which $Q_1 \cap Z(R) \not\subset Q_2 \cap Z(R)$. Then there are infinitely many primes P of R properly contained in Q_1 such that $P \not\subset Q_2$ and $\text{ht } P = 1$.

PROOF. Let TR be the central extension of R obtained by adjoining to R all coefficients of characteristic polynomials of elements of R (which coefficients lie in the quotient ring of R). By Theorem 7 of [1] TR is also Noetherian. By Lemma 1 and Theorem 2 of [7] the extension is integral; hence GU, LO, and INC hold by Theorem 1 of the same paper. It follows that there exists a prime ideal Q'_1 of TR such that $Q'_1 \cap R = Q_1$ and $\text{ht } Q'_1 > 1$. Let $Q'_{21}, Q'_{22}, \dots, Q'_{2k}$ be the set of primes of TR which lie over Q_2 (necessarily of finite number since TR is Noetherian); and let $A = Q'_{21} \cup Q'_{22} \cup \dots \cup Q'_{2k}$. Since $Q_1 \cap Z(R) \not\subset Q'_{2j}$, $1 \leq j \leq k$, it follows that $Q'_1 \cap Z(TR) \not\subset Q'_{2j} \cap Z(TR)$. Let $q_1, q_2, \dots, q_n$ be any finite collection of $\text{ht } 1$ primes of TR lying inside Q'_1 but not in A . TR is integral over $Z(TR)$ [1] and $q_i \neq Q'_1$; so $q_i \cap Z(TR) \neq Q'_1 \cap Z(TR)$, $1 \leq i \leq n$. Then $Q'_1 \cap Z(TR) \not\subset q_1 \cup \dots \cup q_n \cup A$ [5].

Now for any $a \in Q'_1 \cap Z(TR) \setminus (A \cup q_1 \cup \dots \cup q_n)$ there is a $\text{ht } 1$ prime q of TR such that $a \in q \subset Q'_1$ [2]. Obviously, $q \neq q_i$, $1 \leq i \leq n$. Therefore no finite collection of $\text{ht } 1$ primes of TR which lie inside Q'_1 but outside A can exhaust the collection $\{q_r\}$ of such primes. For each r , $q_r \cap R$ is a prime ideal of R properly contained in Q_1 by INC. If $q_r \cap R \subset Q_2$, then $q_r \subset Q'_{2j}$ for some j by GU. This would contradict the assumption that $q_r \not\subset A$. Thus to finish the Lemma, we need only show that at most finitely many of the ideals $q_r \cap R$ can have $\text{ht} > 1$.

Consider in R the ideal $I = \{r \in R | rTR \subset R\}$. Since TR is finitely generated as an R -module by elements of the quotient field of $Z(R)$, $I \neq (0)$. It is not hard to see that a prime ideal of R not containing all of I must be unbranched in TR . Since the primes $\{q_r\}$ have $\text{ht } 1$ in the Noetherian ring TR , at most finitely many can contain I . Hence, at most a finite number of the primes $q_r \cap R$ can contain I . A simple application of GU and LO shows that $I \not\subset q_r \cap R$ implies $\text{ht}(q_r \cap R) = 1$. Thus the Lemma is proved.

THEOREM. *Let $R \subset R'$ be an extension of rings where R is a Noetherian prime PI ring and R' lies in the central integral closure of R . Then $R \subset R'$ has GD and prime ideals of R of $\text{ht} > 1$ are weakly unbranched in R' if and only if prime ideals of R of $\text{ht} > 1$ are unbranched in R' .*

PROOF. If primes of R of $\text{ht} > 1$ are unbranched in R' , it is trivial that $R \subset R'$ has GD since it has LO and GU [7]. Also, as was pointed out above, unbranchedness always implies weak unbranchedness.

In the other direction, suppose that $R \subset R'$ has GD and that $\text{ht} > 1$ primes of R are weakly unbranched in R' . Let Q_1 and Q_2 be two primes of R' chosen such that Q_1 and Q_2 lie over a given prime ideal P of R where $\text{ht } P > 1$. Note that GD forces both Q_1 and Q_2 to be of $\text{ht} > 1$. By hypothesis $Q_1 \cap Z(R') = Q_2 \cap Z(R')$ would imply $Q_1 = Q_2$. We thus assume that $Q_1 \cap Z(R') \subsetneq Q_2 \cap Z(R')$.

Choose $a \in Q_1 \cap Z(R') \setminus Q_2$ and let $S = R[a]$. Since a is integral over R , S is a finite R -module and hence Noetherian. There are only finitely many $\text{ht } 1$ primes of S which contain the ideal $I = \{r \in R \mid rS \subset R\}$. Let $Q'_1 = Q_1 \cap S$ and $Q'_2 = Q_2 \cap S$. By the Lemma there exists a prime P_1 of S such that $I \not\subset P_1$, $P_1 \subsetneq Q'_1$, and $P_1 \not\subset Q'_2$. Observe that $R \subset S$ has GD since $S \subset R'$ has LO, being an integral extension. Since $P_1 \cap R \subset P = Q'_2 \cap R$, it follows by GD that there is a prime $P_2 \subset Q'_2$ such that $P_2 \cap R = P_1 \cap R$. But this could only be true if either $P_1 = P_2$ or $I \subset P_1$. Since both of these possibilities contradict the choice of P_1 , we must conclude that $Q_1 \cap Z(R') = Q_2 \cap Z(R')$. Therefore, P is unbranched in R' .

EXAMPLE 2. Let $J_{(2)}$ be the integers localized at $(2) = 2J$, $R = J_{(2)}[\sqrt{-3}] = \{(a + b\sqrt{-3})/c \mid a, b, c \in J, c \text{ odd}\}$, and

$$P = (2, 1 + \sqrt{-3}) \\ = \{(a + b\sqrt{-3})/c \mid a, b, c \in J, c \text{ odd}, a \text{ and } b \text{ both odd or both even}\}.$$

R is a Noetherian integral domain with unique maximal ideal P . Let $u = (1 + \sqrt{-3})/2$ and $S = R[u] = \{(a + b\sqrt{-3})/2c \mid c \text{ odd}, a \text{ and } b \text{ both odd or both even}\}$. Since $u^2 - u + 1 = 0$, $R \subset S$ is an integral extension. Also, $P = 2S$ is the unique maximal ideal of S . This extension was suggested to the author by Dennis Estes.

To find a counterexample to the noncommutative version of McAdam's theorem, we need prime ideals with $\text{ht} > 1$. So we adjoin an indeterminate and localize again. Let

$$S' = S[x]_{(x,2)} = \{p(x)/q(x) \mid p(x), q(x) \in S[x], q(0) \notin P\}$$

and let

$$I = (x, 2)S' = \{p(x)/q(x) \mid p(x), q(x) \in S[x], q(0) \notin P, p(0) \in P\},$$

the unique maximal ideal of S' . Now take the subring $R' = R[x]_{(x,2,1+\sqrt{-3})} + I$. To see that R' is a proper subring of S' , suppose $u = p(x)/q(x) + r(x)/s(x) \in R'$ where $p(x), q(x) \in R[x]$; $r(x), s(x) \in S[x]$; $q(0), s(0) \notin P$; and $r(0) \in P$. Then setting $x = 0$ gives $u = p(0)q(0)^{-1} + r(0)s(0)^{-1} \in R + P = R$. Since $u \in S \setminus R$, we conclude that $u \in S' \setminus R'$, and the subring is proper. Because I is the unique

maximal ideal of S' , it follows that $S' \setminus R'$ consists of invertible elements. The prime ideal I has $\text{ht} > 1$ since $(0) \subset (x) \subset I$, and these inclusions are proper. It is straightforward to check that $S' = R'[u]$.

For our counterexample we now move into matrix rings. Let

$$A = \left\{ \begin{bmatrix} au + b + e & c \\ d & au^{-1} + b + f \end{bmatrix} \mid a, b \in R', c, d, e, f \in I \right\}.$$

It is easy to check that A is a ring (noting that $u^{-1} = 1 - u$). The ideal $M = \{ \begin{bmatrix} i & j \\ k & l \end{bmatrix} \mid i, j, k, l \in I \}$ is the unique maximal ideal of A . For if $C \in A \setminus M$, then $(\det C)^{-1} \in R'$; so $C^{-1} \in A$. Let $B = A[u]$. Here we identify u with the scalar matrix $\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$. Since $u + u^{-1} = 1$ and

$$\begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -2u & 0 \\ 0 & 0 \end{bmatrix}$$

are in A ,

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} + \begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix} + \begin{bmatrix} -2u & 0 \\ 0 & 0 \end{bmatrix}$$

is in B . Similarly, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in B$. Using these matrix units, one can show that

$$B = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, d \in S'; b, c, d \in I \right\}.$$

B has two maximal ideals,

$$P_1 = \left\{ \begin{bmatrix} s & b \\ c & d \end{bmatrix} \mid s \in S'; b, c, d \in I \right\} \quad \text{and} \quad P_2 = \left\{ \begin{bmatrix} a & b \\ c & s \end{bmatrix} \mid s \in S'; a, b, c \in I \right\}.$$

Observe that $B/P_i \cong S'/I$, a field, $i = 1, 2$. We now make several observations regarding the central extension $A \subset B$.

1. $A \cap P_i = M$ and $P_i \cap Z(B) = \{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \mid a \in I \}$ for $i = 1, 2$. Hence, the ideal M of A is not weakly unbranched in B .

2. The extension is an integral extension of prime PI rings. Since both rings are subrings of 2×2 matrices over the commutative ring S' , they satisfy the standard identity S_4 [6, p. 22]. Both rings are prime since they are orders in the 2×2 matrix ring over the quotient field of S' . Since u is integral over A and in the quotient field of $Z(A) = R'$, B lies in the central integral closure of A . It follows that the extension $A \subset B$ has GU, LO, and INC.

3. Suppose L is an ideal of B and L has an element of the form $\begin{bmatrix} * & * \\ * & s \end{bmatrix}$ with $s \in S' \setminus I$. Then

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} * & * \\ * & s \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix} \in L;$$

hence,

$$\begin{bmatrix} 0 & 0 \\ 0 & s^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in L.$$

It follows that $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix} \in L$ and $\begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in L$ for all $i \in I$. If $i, j \in I$, then $\begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ j & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ j & 0 \end{bmatrix} \in L$. Thus

$$P_2^2 \subset \left\{ \begin{bmatrix} i & j \\ k & s \end{bmatrix} \mid i \in I^2, j, k \in I, s \in S' \right\} \subset L.$$

If L is prime, it must be that $L = P_2$. Similarly, if L contains an element of the form $\begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix}$, $s \in S' \setminus I$, and L is prime, then $L = P_1$. So if L is any prime of B different from P_1 and P_2 , then $L \subset \begin{bmatrix} I & I \\ I & I \end{bmatrix} = M = P_1 \cap P_2 = P_1 \cap A = P_2 \cap A$ —the unique maximal ideal of A . Therefore the extension $A \subset B$ has GD.

4. Since R' and S' are Noetherian, A and B are Noetherian also. The ring A is integral over its center R' and $\text{ht } I > 1$. It follows that the maximal ideal M of A has $\text{ht } > 1$. By Jategaonkar's result in [2] no prime ideal of $\text{ht } > 1$ can be minimal over any central element. Since $Z(A) \subset A$ has INC, $M \cap Z(A) \neq Q \cap Z(A)$ for any $\text{ht } 1$ prime Q of A . Thus there must be infinitely many $\text{ht } 1$ primes of A inside M . By observation 3, each of these $\text{ht } 1$ primes is also a prime ideal of B . Trivially, then, all $\text{ht } 1$ primes of A are unbranched in B .

5. We see that GD does not imply that $\text{ht } > 1$ prime ideals are unbranched in extensions lying within the central integral closure of a Noetherian prime PI ring. Moreover, it is possible for two primes of $\text{ht } > 1$ to contain all smaller primes in their intersection. This can never happen in a commutative Noetherian ring by Krull's Principal Ideal Theorem (see, for example, the argument in [4]). As the proof of the Lemma shows, this seems to be the essential difference between the commutative and noncommutative cases.

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DEPARTMENT OF MATHEMATICS, BRYAN COLLEGE, DAYTON, TENNESSEE 37321