ON MONOGENIC OPERATORS AND MEASURES

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ABSTRACT. The notion of a monogenic operator between linear lattices, generalizing that of a monogenic measure, is introduced and investigated. The decomposition of an operator into its monogenic and antimonogenic parts is established. Products of monogenic measures are also considered.

Introduction. We introduce the notion of a monogenic operator between linear lattices, i.e. of an order bounded linear operator which is determined in some sense by its restriction to a given linear sublattice. We prove that monogenic operators form a band (Theorem 1). This yields the decomposition of an arbitrary operator into its monogenic and antimonogenic parts. In a special case where operators are identified with real-valued measures on a σ -field we get a generalization of a result of Johnson [4]. We compare our decomposition with some classical decompositions in measure theory (Examples 1 and 2). We also consider (not necessarily direct) products of monogenic measures (Theorem 2 and Example 3). As far as the linear-lattice-theoretical terminology is concerned we mostly follow Jameson's book [3].

1. Monogenic operators. Let X and Y be real linear lattices and assume that Y is order complete. Then the set $L_b(X, Y)$ of order bounded linear operators from X into Y is an order complete linear lattice [3, Theorem 2.6.1]. We denote by $L_+(X, Y)$ its subset consisting of all positive (i.e. monotonic in the terminology of [3]) operators. Let M be a cofinal (i.e. majorizing) linear sublattice of X and $\mathcal{L}$ be a solid subspace of $L_b(X, Y)$. We say that $T \in \mathcal{L}$ is *monogenic* (with respect to M and $\mathcal{L}$) provided $|T|M| = |T| \cdot |M|$ and there is no other $S \in \mathcal{L}$ with this property which coincides with T on M (more precisely, $|S|M| = |S| \cdot |M|$ and $S|M| = T|M|$ imply $S = T$). In case T is positive, this definition can be made simpler. Moreover, alternate descriptions of monogenicity are available.¹ Namely we have

LEMMA 1. Suppose $T \in \mathcal{L}_+$. Then the following four conditions are equivalent:

- (i) T is monogenic.
- (ii) For any $S \in \mathcal{L}_+$ with $S|M| = T|M|$ we have $S = T$.
- (iii) For any $S \in \mathcal{L}_+$ with $S|M| \leq T|M|$ we have $S \leq T$.
- (iv) For any $S \in \mathcal{L}_+$ with $S|M| \geq T|M|$ we have $S \geq T$.

Received by the editors June 15, 1979.

AMS (MOS) subject classifications (1970). Primary 28A10, 28A35, 47B55.

Key words and phrases. Linear lattice, band, monogenic operator, monogenic measure, products of measures.

¹Note that for $\mathcal{L} = L_b(X, Y)$ an intrinsic characterization of positive monogenic operators is possible (see [5, Theorem 1]).

PROOF. Clearly, (i) $\Rightarrow$ (ii). To prove the converse implication, it is enough to show that if $S \in L_b(X, Y)$ and $S|M = |S|M$, then S is positive. Let $x \in X_+$ and take $z \in M$ with $x < z$. Then $|x - z| < z$. It follows that $S(x) = |S|(z) + S(x - z) > 0$. Next we prove that (ii) $\Rightarrow$ (iii). Let S satisfy the assumptions of (iii). By Theorem 2.6.3 in [3] applied to the lattice-monotonic sublinear mapping $P(x) = T(|x|)$, there exists $S_1 \in L_+(X, Y)$ with $S_1|M = S|M$ and $S_1 < T$. Hence $((T - S_1) + S)|M = T|M$ and $T - S_1 + S \in \mathcal{F}_+$, so that in view of (ii), $S = S_1$, and we are done. A similar, but simpler argument yields (ii) $\Rightarrow$ (iv). Finally the implications (iii) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (ii) are an easy consequence of the following

LEMMA 2. If $S, T \in L_+(X, Y)$, $S < T$ and $S|M = T|M$, then $S = T$.

PROOF. Given $x \in X$ choose $z \in M$ with $x < z$. Then $S(x) = S(z) - S(z - x) = T(z) - S(z - x) > T(z) - T(z - x) = T(x)$.

THEOREM 1. The set of monogenic operators is a band in $\mathcal{F}$.²

PROOF. First we show that if $T \in \mathcal{F}$ is monogenic, then so is $|T|$. Take $W \in \mathcal{F}_+$ with $|T||M = W|M$. In view of Lemmas 1 and 2, it is enough to show that $|T| < W$. By the generalized version of the Hahn-Banach theorem [3, 2.5.7], there exists a linear operator $T_1: X \rightarrow Y$ with $T_1|M = T|M$ and $|T_1(x)| < W(|x|)$ for $x \in X$. Hence $|T_1| < W$ and $T_1 \in \mathcal{F}$. As T is monogenic, we have $|T_1|M < W|M = |T|M = |T_1|M$, and so $T_1 = T$. Thus $|T| < W$.

In order to establish the converse assertion suppose $T \in \mathcal{F}$ and $|T|$ is monogenic, and take $S \in \mathcal{F}$ with $|S|M = |S||M$ and $S|M = T|M$. Then $|S||M < |T||M$, whence by Lemma 1, $|S| < |T|$. It follows that for any $x \in X$ and $z \in Z$ we have $|S(x) - T(x)| = |S(x - z) - T(x - z)| < 2|T|(|x - z|)$. Hence, in view of Theorem 3 in [5], $S(x) = T(x)$.

Suppose $T_1, T_2 \in \mathcal{F}_+$ are monogenic. We claim that $T_1 + T_2$ is also monogenic. According to Lemma 1, it is enough to show that given $S \in \mathcal{F}_+$ with $(T_1 + T_2)|M < S|M$ we have $T_1 + T_2 < S$. The latter follows easily by a repeated application of Lemma 1.

Clearly, tT is monogenic for any $T \in \mathcal{F}$ and $t \in \mathbb{R}$. It is also easily seen from Lemma 1 that if $T \in \mathcal{F}_+$ is monogenic, then so is any $S \in L_+(X, Y)$, with $S < T$. These observations together with what we have proved so far show that the set of monogenic operators is a solid subspace of $\mathcal{F}$. Thus in order to prove that it is a band, it is enough to note that if $T_\alpha \in \mathcal{F}_+$ are monogenic and $T = \sup T_\alpha$ (in $\mathcal{F}$), then T is also monogenic [3, 2.4.3]. But this is so as T satisfies condition (iv) of Lemma 1.

We say that $T \in \mathcal{F}$ is *antimonogenic* (with respect to M and $\mathcal{F}$) provided T is disjoint from all monogenic operators in $\mathcal{F}$. Theorem 1 and the Riesz decomposition theorem [3, 2.5.4] yield the following

²Dr. S. S. Kutateladze has kindly informed us that a result related to Theorem 1 can be found in his paper *Choquet boundaries in K-spaces*, *Uspehi Mat. Nauk* **30** (1975), 107–146, Theorem 3.1; English transl. in *Russian Math. Surveys* **30** (1975), 115–155.

COROLLARY 1. *Suppose $\mathcal{J}$ is a band in $L_b(X, Y)$. Then each $T \in \mathcal{J}$ can be uniquely decomposed in the following way: $T = T_1 + T_2$, where $T_1, T_2 \in \mathcal{J}$, T_1 is monogenic and T_2 is antimonogenic.*

2. Monogenic measures. Let Σ be a σ -field of subsets of a set Ω and denote by $s(\Sigma)$ the linear lattice of Σ -measurable simple functions. The linear lattice $ba(\Sigma)$ of bounded real-valued finitely additive set functions on Σ can be identified with $L_b(s(\Sigma), \mathbb{R})$ (cf. [5, Section 2]). Under this identification the set $ca(\Sigma)$ of real-valued measures on Σ becomes a band in $L_b(s(\Sigma), \mathbb{R})$.

Let Σ' be a sub- σ -field of Σ . We say that $\mu \in ca(\Sigma)$ is *monogenic* (*antimonogenic*) with respect to Σ' if it is a monogenic (antimonogenic) operator with respect to $s(\Sigma')$ and $ca(\Sigma)$. It is easily seen that $\mu \in ca(\Sigma)$ is monogenic if and only if $|\mu|_{\Sigma'} = |\mu|_{\Sigma'}$ and there is no other $\nu \in ca(\Sigma)$ with this property which coincides with μ on Σ' . It is also clear that $\mu \in ca_+(\Sigma)$ is monogenic if and only if for any $\nu \in ca_+(\Sigma)$ with $\nu|_{\Sigma'} = \mu|_{\Sigma'}$ we have $\nu = \mu$ (cf. Lemma 1). This shows that our definition generalizes (in two directions) the one given by Berberian [2, p. 231] and considered also by Johnson [4, §3].

Clearly $\mu \in ca_+(\Sigma)$ is monogenic whenever Σ is contained in the completion of Σ' with respect to $\mu|_{\Sigma'}$. By a result of Łoś-Marczewski [7, Theorem 4] this condition is also necessary if Σ is generated by $\Sigma' \cup \{A\}$, where $A \subset \Omega$. Hence, in view of a theorem of Bierlein [1, Corollary 3] this subsists if A is replaced by an arbitrary family of disjoint subsets of Ω . Due to a theorem of Varadarajan [13, Corollary to Lemma 2.2]; see also [1, Corollary 4], the same is true in case Ω is an analytic subset of a Polish space, Σ is its Borel σ -field and Σ' is an arbitrary σ -generated sub- σ -field of Σ . However, if Σ is "much larger" than Σ' it may happen that μ is monogenic even though $\mu|_{\Sigma'}$ is complete (see [10] or [11]).

As a special case of Theorem 1 we get

COROLLARY 2.³ *The set of monogenic measures is a band in $ca(\Sigma)$.*

The next corollary generalizes a result of Johnson [4, Theorem 3.3].

COROLLARY 3. *Each $\mu \in ca(\Sigma)$ can be uniquely decomposed in the following way: $\mu = \mu_1 + \mu_2$, where $\mu_1, \mu_2 \in ca(\Sigma)$, μ_1 is monogenic and μ_2 is antimonogenic. In particular, if $\mu \ll \nu$, where $\nu \in ca(\Sigma)$ is monogenic, then so is μ .*

PROOF. As $ca(\Sigma)$ is a band in $ba(\Sigma)$ [6, Example 25.3], the first assertion is a special case of Corollary 1. The second assertion follows from this since $\mu_2 \ll \nu$ and $|\mu_2| \wedge |\nu| = 0$ imply $\mu_2 = 0$ (cf. [6, p. 144, (ii)]).

We shall illustrate Corollary 3 by two examples.

EXAMPLE 1. Let Σ be the Borel σ -field of the real line and let Σ' be its sub- σ -field consisting of all countable sets and their complements. Then a measure on Σ is

³Let us note that in the case of measures the essential implications (ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv) of Lemma 1 can also be proved with the help of the Radon-Nikodym theorem.

monogenic (resp., antimonogenic) if and only if it is discrete (resp., continuous, i.e., vanishing on points). Thus the decomposition of Corollary 3 coincides in this case with the decomposition of a Borel measure into its discrete and continuous parts.

Before passing to the other example let us note that if Ω is an uncountable Borel subset of a Polish space and μ is a (nonzero) continuous universal measure on Ω , then μ is not monogenic with respect to the Borel σ -field $B(\Omega)$ of Ω . Indeed, assume μ is a probability measure (cf. Corollary 2). For $\Omega = [0, 1]$ and μ being an extension of Lebesgue measure the assertion follows from the well-known fact that μ is not invariant. In general, $\mu|B(\Omega)$ is pointwise isomorphic to Lebesgue measure on $B([0, 1])$ ([8, Theorem 4.1 (ii)], or [12, p. 327]), so a reduction to the previous case is possible.

EXAMPLE 2. Let Ω be the real line again, let ν_0 be the completion of a continuous Borel measure on Ω and let Σ denote the domain of ν_0 . Choose Σ' as the Borel σ -field of Ω . It turns out then that in case $\mu \in ca(\Sigma)$ is continuous Corollary 3 yields the Lebesgue decomposition of μ relative to ν_0 . This is because a continuous measure $\nu \in ca(\Sigma)$ is monogenic if and only if $\nu \ll \nu_0$. One implication follows from Corollary 3. To prove the other assume that $\nu \ll \nu_0$ fails. Then there is $\Omega_0 \in \Sigma'$ with $|\nu_0|(\Omega_0) = 0$ and $|\nu|(\Omega_0) > 0$. It follows that ν restricted to $\Omega_0 \cap \Sigma$ is a continuous universal measure on Ω_0 , and so it is not monogenic with respect to the Borel- σ -field of Ω_0 ($= \Omega_0 \cap \Sigma$) (see the passage following Example 1). Hence ν is not monogenic.

In the sequel we shall restrict attention to positive measures. From the previous results we shall only need the obvious assertion that a positive measure majorized by a monogenic measure is itself monogenic. The following is a generalization of a result of Johnson [2, Exercise 70.14].

THEOREM 2. Let, for $i = 1, 2$, Σ_i and Σ'_i be σ -fields of subsets of a set Ω_i with $\Sigma'_i \subset \Sigma_i$. Suppose $\mu_i \in ca_+(\Sigma)$ and $\mu \in ca_+(\Sigma_1 \otimes \Sigma_2)$ is a product of μ_1 and μ_2 .

(a) If μ_i is monogenic with respect to Σ'_i , $i = 1, 2$, then μ is monogenic with respect to $\Sigma'_1 \otimes \Sigma'_2$.

(b) If μ is the direct product of μ_1 and μ_2 and μ is monogenic with respect to $\Sigma'_1 \otimes \Sigma'_2$, then μ_i is monogenic with respect to Σ'_i , $i = 1, 2$.

PROOF. Assertion (b) is obvious. To prove (a) suppose $\nu \in ca_+(\Sigma)$ and $\nu|_{\Sigma'_1 \otimes \Sigma'_2} = \mu|_{\Sigma'_1 \otimes \Sigma'_2}$. It follows that $\nu(F_1 \times F_2) = \mu(F_1 \times F_2) \leq \mu_1(F_1)$ for $F_i \in \Sigma'_i$. Fixing F_2 for a moment, we infer from the monogenicity of μ_1 that $\nu(E_1 \times F_2) = \mu(E_1 \times F_2)$ for all $E_1 \in \Sigma_1$. A similar argument yields $\nu(E_1 \times E_2) = \mu(E_1 \times E_2)$ for all $E_i \in \Sigma'_i$. Hence $\nu = \mu$.

Assertion (b) fails for nondirect products. Here is an example to this effect.

EXAMPLE 3. By [9], (ii) there exist probability spaces $(\Omega_i, \Sigma'_i, \mu'_i)$, $i = 1, 2$, a product μ' of μ'_1 and μ'_2 , and a set $A \subset \Omega_1$ such that its inner μ'_1 -measure equals 0 while the inner μ' -measure of $A \times \Omega_2$ equals 1. Let Σ_1 be the σ -field generated by $\Sigma'_1 \cup \{A\}$ and $\Sigma_2 = \Sigma'_2$. Then μ' extends uniquely to a measure $\mu \in ca_+(\Sigma_1 \otimes \Sigma_2)$ but μ'_1 can be extended in different ways to a positive measure on Σ_1 .

ADDED IN PROOF. The same example as in [10], [11] has also been given by Z. Riečanová, *A monogenic Baire measure need not be completion regular*, Mat. Časopis Sloven. Akad. Vied **24** (1974), 275–276.

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