INEQUALITIES FOR DERIVATIVES OF POLYNOMIALS WITH RESTRICTED ZEROS¹

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ABSTRACT. Let f be a polynomial of degree n that has real roots all of which lie outside the interval $(-1, 1)$. P. Erdős proved that if $|f| < 1$ on this interval then $|f'| < ne/2$ on $[-1, 1]$; this is much stronger than the results the inequalities of A. Markov and S. N. Bernstein would give. We will show that if only $n - k$ roots of f are restricted as above, then $|f'| < c_k n$ holds on $[-1, 1]$ with appropriate c_k . An upper estimate for the best c_k is given. Results for higher derivatives and L^p spaces are also obtained.

1. Introduction and statement of results. Let $-1 < x < 1$ in what follows. P. Erdős [1] proved that if f is a polynomial of degree n with real roots all of which lie outside the interval $(-1, 1)$, then

$$|f'(x)| < M_f ne/2, \quad (1)$$

where M_f denotes the maximum of $|f|$ on the interval $[-1, 1]$. This is to be compared with the well-known inequalities on A. Markov and S. N. Bernstein,

$$|f'(x)| < M_f n^2 \quad \text{and} \quad |f'(x)| < M_f n(1 - x^2)^{-1/2},$$

respectively, which are true without any assumptions about the roots of f (cf. e.g. [5]). P. Erdős asked if an inequality similar to (1) remains valid if k roots of f are allowed to lie inside $(-1, 1)$, i.e. whether it is then true with an appropriate constant c_k that

$$|f'(x)| < M_f n c_k. \quad (2)$$

This was recently shown by J. Szabados and A. K. Varma for $k = 1$ [6]. The motivation behind the above question is partly that (1) cannot be iterated to estimate higher derivatives since e.g. f' may have one root in $(-1, 1)$. Here we will establish (2) and obtain an upper estimate for the best possible c_k . It will turn out that higher derivatives can be estimated more precisely in a direct way than by iterating (2).

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Write $\mathcal{P}(n, k)$ for the set of polynomials of degree n that have $n - k$ real roots outside the interval $(-1, 1)$ (the remaining k roots can be arbitrary complex numbers). Let $n > k > 0$. We will prove that

$$|f'(x)| < 6ne^{\pi\sqrt{k}} M_f \quad (3)$$

holds for any $f \in \mathcal{P}(n, k)$. For higher order derivatives we will show that

$$|f^{(l)}(x)| < 3\sqrt{l} \, 6^n l' \exp(-l + \pi\sqrt{kl}) M_f, \quad (4)$$

where $f \in \mathcal{P}(n, k)$. For $k = 0$ we have a somewhat stronger result,

$$|f^{(l)}(x)| < 3\sqrt{l} \, (6/e)^l n^l |f(x \pm l/(2n))|, \quad (5)$$

where the sign $+$ or $-$ is taken according as $x < 0$ or $x > 0$ (either may be taken for $x = 0$). As we may assume $l < n$, we have $-1 < x \pm l/(2n) < 1$, i.e. $|f(x \pm l/(2n))| < M_f$. (5) can be used to derive a result for L^p spaces (see (24)). If n^2/l^3 is large, (3)–(5) can be slightly strengthened as

$$|f'(x)| < 2n \exp(1 + \pi\sqrt{k})(1 + O(l^3 n^{-2}d)) M_f, \quad (6)$$

where the function indicated by $O(\cdot)$ does not depend on k ,

$$|f^{(l)}(x)| < 3\sqrt{l} \, 2^n l' \exp(\pi\sqrt{kl}) M_f, \quad (7)$$

and

$$|f^{(l)}(x)| < 3\sqrt{l} \, 2^n l' |f(x \pm l/(2n))|, \quad (8)$$

respectively. We do not know how accurate these estimates are.

2. We start out with the relatively simple

PROOFS OF (5) AND (8). Write

$$f(x) = \prod_{j=1}^n (x - x_j), \quad (9)$$

where $x_j < -1$ for $1 < j < m$ and $x_j > 1$ for $m < j < n$. To estimate $f^{(l)}(x)$, suppose e.g. that $-1 < x < 0$, and let C be the circle with center x and radius $l/(2n)$ in the complex plane. Then, according to Cauchy's formula, we have

$$\begin{aligned} |f^{(l)}(x)| &= \left| l! (2\pi i)^{-1} \oint_C f(z) / (z - x)^{l+1} dz \right| \\ &< l! (2n/l)^l \max\{|f(z)|: z \in C\}. \end{aligned} \quad (10)$$

Fix $z \in C$ and write $x_0 = x + l/(2n)$. We are going to estimate $|f(z)/f(x_0)|$ with the aid of representation (9). We have

$$|z - x_j|/|x_0 - x_j| < 1 \quad (11)$$

for $1 < j < m$, and

$$|z - x_j|/|x_0 - x_j| < (1 + l/(2n))/(1 - l/(2n)) \quad (12)$$

for $m < j < n$, as we assumed $x < 0$. Thus

$$|f(z)/f(x_0)| < ((1 + l/(2n))/(1 - l/(2n)))^n < 3' \quad (13)$$

for $l \leq n$, which we may assume. Now (5) follows from (10)–(12) and the form

$$l! < (1 + 1/(l-1))\sqrt{2\pi} l^{l+1/2} e^{-l} < 3l^{l+1/2} e^{-l} \quad (14)$$

of Stirling's formula, valid for any $l \geq 1$ (see e.g. [2, pp. 52–54]). (8) also follows if one notes that in (13) the right-hand side can be replaced with $e^l(1 + O(l^3 n^{-2}))$; there is no need to include the factor multiplying e^l here on the right-hand side of (8) in view of the crudeness of the estimate in (14).

3. For the proofs of the remaining relations we will need the following

LEMMA. Let h be a polynomial of degree $k \geq 1$ and let l be any positive real. Then there is a t with $\exp(-\pi\sqrt{k/l}) < t < 1$ such that

$$|w^{-l}h(w)/h(|w|)|\exp(\pi\sqrt{kl}) \quad (15)$$

holds for any complex w with $|w| = t$.

PROOF. Write $h(w) = \prod_{j=1}^k (w - w_j)$ and $p(w) = \prod_{j=1}^k (w + |w_j|)$. Then, clearly,

$$|h(w)/h(|w|)| < |p(|w|)/p(-|w|)|.$$

Hence, to establish (15), we have to prove only that there is a t in the range described above for which

$$|t^{-l}p(t)/p(-t)| < \exp(\pi\sqrt{kl}) \quad (16)$$

holds. For $l = 1$, this is essentially the same as Lemma 4 in [4, p. 13]; the case of arbitrary positive l can be handled similarly. For the convenience of the reader, we include the details. If we assume that (16) fails, then, writing $\delta = \exp(-\pi\sqrt{k/l})$, we obtain

$$\int_{\delta}^1 t^{-1} \log |t^{-l}p(t)/p(-t)| dt \geq \pi\sqrt{(kl)} \int_{\delta}^1 t^{-1} dt = \pi^2 k. \quad (17)$$

On the other hand, writing $\xi_j = |w_j|$, we have

$$\begin{aligned} & \int_{\delta}^1 t^{-1} \log |t^{-l}p(t)/p(-t)| dt \\ &= \int_{\delta}^1 -lt^{-1} \log t dt + \sum_{j=1}^k \int_{\delta}^1 t^{-1} \log |(t + \xi_j)/(t - \xi_j)| dt < \pi^2 k. \end{aligned} \quad (18)$$

To see the last inequality here, note that the first term in the expression immediately preceding the inequality sign equals $\pi^2 k/2$, as is easily seen; and, as for the second term, we have, for any $\xi > 0$,

$$\begin{aligned} & \int_{\delta}^1 t^{-1} \log |(t + \xi)/(t - \xi)| dt = \int_{\delta/\xi}^{1/\xi} t^{-1} \log |(t + 1)/(t - 1)| dt \\ & < \int_0^{\infty} t^{-1} \log |(t + 1)/(t - 1)| dt = 2 \int_0^1 t^{-1} \log ((1 + t)/(1 - t)) dt \\ &= 2 \int_0^1 2(1 + t^2/3 + t^4/5 + \dots) dt = 4(1 + 1/3^2 + 1/5^2 + \dots) = \pi^2/2. \end{aligned}$$

The contradiction between (17) and (18) establishes (16). The proof of the Lemma is complete.

Inequalities (3), (4), (6), and (7) are direct consequences of the following

THEOREM 1. *Let k, l , and n be integers with $1 < k < n$ and $1 < l < n$; assume that $f \in \mathcal{P}(n, k)$. Write*

$$M_f(x) = \max\{|f(x \pm r)|: l \exp(-\pi\sqrt{k/l})/(2n) < r \leq l/(2n)\}, \quad (19)$$

where the sign $+$ or $-$ is taken according as $x \leq 0$ or $x > 0$ (either may be taken for $x = 0$). Then (3), (4), (6), and (7) remain valid if M_f is replaced by $M_f(x)$ in them (we will call these modified formulas (3'), (4'), (6'), and (7'), respectively).

PROOF. Let f again be represented as in (9), where we now assume that $x_j \leq -1$ for $k < j \leq m$ and $x_j > 1$ for $m < j \leq n$, while x_j is arbitrary for $1 \leq j \leq k$. To estimate $f^{(l)}(x)$, suppose e.g. that $-1 \leq x \leq 0$. Write $r = lt/(2n)$, where $\exp(-\pi\sqrt{k/l}) < t < 1$, and let C_r be the circle with center x and radius r in the complex plane. Similarly as in (10), by Cauchy's formula we have

$$|f^{(l)}(x)| < l! (2n/l)^l t^{-l} \max\{|f(z)|: z \in C_r\}. \quad (20)$$

Write $g(u) = \prod_{j=1}^k (u - x_j)$ and $h(w) = g(x + lw/(2n))$. According to (15), there is a t_0 with $\exp(-\pi\sqrt{k/l}) < t_0 < 1$ such that we have

$$|w^{-l}h(w)/h(|w|)| < \exp(\pi\sqrt{kl}) \quad (21)$$

for $|w| = t_0$. Put $r_0 = lt_0/(2n)$ and fix $z \in C_{r_0}$; write $x_0 = x + r_0$. Then (21) in effect says that

$$t_0^{-l} |g(z)/g(x_0)| < \exp(\pi\sqrt{kl}). \quad (22)$$

(11) and (12) remain valid except that (11) now holds only for $k < j \leq m$. (22), (11), and (12) imply that

$$t_0^{-l} |f(z)| \leq 3^l \exp(\pi\sqrt{kl}) |f(x_0)| \quad (23)$$

(cf. (13), which is, however, not valid at present). (3'), i.e. the modification of (3) as described in Theorem 1, now follows from (20) with $r = r_0$ and $t = t_0$ and (23) by taking $l = 1$ and observing that $|f(x_0)| \leq M_f(x)$. (4') follows by using Stirling's formula (14) as well. (6') and (7') can be established by observing that 3^l on the right-hand side of (23) can be replaced with $e^l (1 + O(l^3 n^{-2}))$ in the same way as was done with (13) in the proof of (8). Again, there is no need to include the factor multiplying e^l here on the right-hand side of (7') in view of the crudeness of the estimate in (14). This completes the proof of Theorem 1.

4. L^p spaces. An easy consequence of (5) is the following

THEOREM 2. *Let f be a polynomial of degree n all roots of which are real and lie outside $(-1, 1)$. Let $p > 0$. Then*

$$\int_{-1}^1 |f^{(l)}|^p < 2(3\sqrt{l} (6/e)^l n^l)^p \int_{-1}^1 |f|^p. \quad (24)$$

PROOF. Raise (5) to the p th power and integrate. The result then follows by noticing that the expression $x \pm l/(2n)$ assumes the same value at most twice. The limits of the integral on the right-hand side can be replaced with $-1 + l/(2n)$ and $1 - l/(2n)$, respectively.

A feature of (24) to which we would like to call attention is that it covers the case $0 < p < 1$ in particular. Until recently, Bernstein's inequality for L^p spaces concerning trigonometric polynomials (or algebraic polynomials on the unit circle) had been known only for $p \geq 1$; cf. e.g. [7, Vol. II, (3.18), p. 11]. In [3, Theorem 4, p. 148], this inequality was extended for the case $0 < p < 1$.

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