RESULTS ON BI-UNIVALENT FUNCTIONS

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ABSTRACT. When the class σ of bi-univalent functions was first defined, it was known that functions of the form $\phi \circ \psi^{-1} \in \sigma$ when ϕ and ψ are univalent, map the unit disc $\mathbb{B}$ onto a set containing $\mathbb{B}$, and satisfy $\phi(0) = \psi(0) = 0$, $\phi'(0) = \psi'(0)$. It is shown here that such functions form a proper subset of σ , and that σ is a proper subset of the set of functions of the form $\phi \circ \psi^{-1}$, where ϕ and ψ are locally univalent, at most 2-valent, each maps a subregion of $\mathbb{B}$ univalently onto $\mathbb{B}$, and $\phi(0) = \psi(0) = 0$, $\phi'(0) = \psi'(0)$, $\psi^{-1}(0) = 0$. It is also shown that there are $f(z) = z + a_2 z^2 + \dots$ in σ with $|a_2| > 4/3$. However, doubt is cast that $|a_2|$ can be as large as $3/2$.

1. Introduction. Let S be the class of normalized univalent functions $f(z) = z + a_2 z^2 + \dots$, and let $\mathbb{C}$ denote the complex plane. In 1967, Lewin [2] defined the class σ of bi-univalent functions as follows: $h \in \sigma$ if $h \in S$ and $h^{-1} \in S$, i.e., h^{-1} has an analytic extension through the unit disc, $\mathbb{B}$, which is univalent on $\mathbb{B}$. Through sophisticated use of the Grunsky inequalities, Lewin showed that $|a_2| < 1.51$.

Let Λ_1 be the class of functions $\phi(z) = A_1 z + A_2 z^2 + \dots$ which are univalent in $\mathbb{B}$ and satisfy $\phi(\mathbb{B}) \supset \mathbb{B}$. Let σ_1 be the collection of functions of the form $\phi \circ \psi^{-1}$ where $\phi, \psi \in \Lambda_1$ and $\phi'(0) = \psi'(0)$. It was pointed out by Nehari (see [2]) that $\sigma_1 \subset \sigma$, but it was not known if $\sigma_1 = \sigma$. Suffridge [4] produced a function in σ_1 with $a_2 = 4/3$ and conjectured that $|a_2| < 4/3$ for all functions in σ . Netanyahu [3] proved that in σ_1 , $|a_2| < 4/3$ is correct.

Let Λ_2 be the class of functions $\phi(z) = A_1 z + A_2 z^2 + \dots$ such that ϕ is locally univalent and at most 2-valent in $\mathbb{B}$, and there is a neighborhood, N , of zero which ϕ maps univalently onto $\mathbb{B}$. Let σ_2 be the collection of functions of the form $\phi \circ \psi^{-1}$ on $\mathbb{B}$ where $\phi, \psi \in \Lambda_2$ and $\phi'(0) = \psi'(0)$. Here the branch of ψ^{-1} is chosen which sends zero to zero. Clearly $\Lambda_1 \subset \Lambda_2$ and $\sigma_1 \subset \sigma_2$.

In this paper we show that $\sigma_1 \subsetneq \sigma \subsetneq \sigma_2$. Recently Bshouty, Hengartner and Schober [1] have independently given a function, similar to our Example 1, showing that $\sigma_1 \neq \sigma$. We also show that there are functions in σ with $a_2 > 4/3$. Our work leads to several open questions.

2. $\sigma_1 \subsetneq \sigma \subsetneq \sigma_2$. We begin with a function in σ which is not in σ_1 .

EXAMPLE 1. For $0 < \theta < \pi/2$ consider the functions $h_\theta \in S$ given by

$$h_\theta(z) = \cos \theta \frac{ze^{-i\theta}}{1 - (ze^{-i\theta})^2} + \sin \theta \left[\frac{i}{2} \log \frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}} \right].$$

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The image of $\mathbf{B}$ under h_θ is the complex plane less two vertical slits $\{w = u_0(\theta) + iv: v < v_0(\theta)\}$ and $\{w = -u_0(\theta) + iv: v > -v_0(\theta)\}$, where $u_0(\theta) = (\pi/4)\sin \theta$ and $v_0(\theta) \rightarrow +\infty$ as $\theta \rightarrow \pi/2^-$. Let θ be sufficiently near $\pi/2$, that $u_0(\theta) > \frac{1}{2}$ and $u_0(\theta) + iv_0(\theta) \notin \mathbf{B}$. Then, by the reflection principle, h_θ^{-1} has an analytic extension from $\mathbf{B} \cap \{w: |\operatorname{Re} w| < u_0(\theta)\}$ to $\mathbf{B}$, which is readily seen to be one-one on $\mathbf{B}$. Thus, $h_\theta \in \sigma$ if θ is sufficiently near $\pi/2$.

Now, suppose $h_\theta \in \sigma_1$, i.e., $h_\theta = \phi \circ \psi^{-1}$ for some $\phi, \psi \in \Lambda_1$ with $\phi'(0) = \psi'(0)$. Since ϕ must map $\psi^{-1}(\mathbf{B})$ onto $h_\theta(\mathbf{B})$, and since $h_\theta(\mathbf{B})$ is dense in the plane, ϕ cannot be one-one on a region larger than $\psi^{-1}(\mathbf{B})$. Thus, $\psi^{-1}(\mathbf{B}) = \mathbf{B}$ and $h_\theta(\mathbf{B}) = \phi(\mathbf{B})$. This is impossible since $\mathbf{B} \subset \phi(\mathbf{B})$, but $\mathbf{B} \not\subset h_\theta(\mathbf{B})$.

This proof shows, in fact, that if $f \in \sigma_1$, then $f(\mathbf{B})$ is not dense in the plane.

In order to show that $\sigma \subset \sigma_2$, we shall construct for $h \in \sigma$ a simply-connected, two sheeted Riemann surface, M , over $\mathbf{C}$ by piecing together $\mathbf{B}$ and $h(\mathbf{B})$ along the component of $h(\mathbf{B}) \cap \mathbf{B}$ which contains zero. We then define $\phi: \mathbf{B} \rightarrow \mathbf{C}$ so that $\phi(0) = 0$ and the Riemann surface for ϕ^{-1} is M . It will follow that $\phi \in \Lambda_2$ and that $\psi = h^{-1} \circ \phi \in \Lambda_2$. The details are given below.

PROPOSITION. $\sigma \subset \sigma_2$.

PROOF. Let $h \in \sigma$, $U_1 = \{(w, 1): w \in \mathbf{B}\}$ and $U_2 = \{(w, 2): w \in h(\mathbf{B})\}$. Further, set $(w, 1) = (\tilde{w}, 2)$ if and only if $w = \tilde{w}$ and this point lies in the component, Ω , of $\mathbf{B} \cap h(\mathbf{B})$ which contains zero. Let $M = U_1 \cup U_2$ with the topology inherited from $\mathbf{C}$. Define $\pi: M \rightarrow \mathbf{C}$ by $\pi(w, j) = w$ and let π_j denote the restriction of π to U_j , $j = 1, 2$. Then $\pi_i \circ \pi_j^{-1}$ is the identity map where it is defined. Thus, $\{(\pi_1, U_1), (\pi_2, U_2)\}$ is an analytic atlas for M , and M is a Riemann surface. Also, M is simply connected since U_1 and U_2 are, and $U_1 \cap U_2$ is connected. Clearly M is hyperbolic, and thus there is an analytic bijection g from $\mathbf{B}$ onto M with $g(0) = (0, 1)$. Define $\phi: \mathbf{B} \rightarrow \mathbf{C}$ by $\phi = \pi \circ g$. Clearly $\phi \in \Lambda_2$.

Since $h \in \sigma$, h^{-1} has an analytic one-one branch on $\mathbf{B}$, which we shall call h_1^{-1} . Further, we denote by h_2^{-1} the one-one analytic branch of h^{-1} on $h(\mathbf{B})$. Define $\psi: \mathbf{B} \rightarrow \mathbf{C}$ by $\psi(z) = (h_j^{-1} \circ \pi \circ g)(z)$ if $g(z) \in U_j$, $j = 1, 2$. It is assured that ψ is well defined since $h_1^{-1} = h_2^{-1}$ on Ω . Clearly $\psi \in \Lambda_2$ and $h = \phi \circ \psi^{-1}$. Thus $h \in \sigma_2$.

We note that ϕ covers Ω exactly once and ψ covers $h^{-1}(\Omega)$ exactly once. Thus, an extremal function for maximizing $|a_2|$ over σ may be found amongst functions in Λ_2 which cover a neighborhood of zero exactly once.

The next example shows that σ is a proper subset of σ_2 .

EXAMPLE 2. For $\alpha > 1$, let $\tau(z) = (z + 1)^\alpha / 2^{(\alpha-1)} - 1$, $\tau(1) = 1$. We show below that there is a neighborhood N of zero such that $\tau|_N$ is one-one and $\tau(N) = \mathbf{B}$. Let x_0 be the zero of τ in N , $g(z) = (z + x_0)/(1 + x_0 z)$, and consider $\phi = \tau \circ g$. For $2 < \alpha \leq 4$, $\phi \in \Lambda_2 \setminus \Lambda_1$. If $\psi(z) = -\phi(-z)$, then $\psi \in \Lambda_2$ and $h = \phi \circ \psi^{-1} \in \sigma_2$. However, h fails to be univalent near -1 when $\alpha > 2$. In particular, for $\alpha = 3$, h is holomorphic at -1 and has there a critical point of order 2. Thus, $h \notin \sigma$.

It remains to show that τ covers $\mathbf{B}$ univalently, or equivalently, that $f(z) = (z + 1)^\alpha$ covers univalently the disc D with center $2^{\alpha-1}$ and radius $2^{\alpha-1}$. Observe that $f(e^{it}) = [2 \cos(t/2)]^\alpha e^{iat/2}$, and consider $\rho = 2^\alpha \cos(\alpha t/2)$, which describes ∂D

in polar form. It suffices to show that $2^\alpha \cos(\alpha t/2) < [2 \cos(t/2)]^\alpha$ for $0 < t < \pi/\alpha$. Letting $\nu(t) = [\cos(t/2)]^\alpha - \cos(\alpha t/2)$, we have $\nu(0) = 0$ and

$$\nu'(t) = \frac{\alpha}{2} \left[\sin\left(\frac{\alpha t}{2}\right) - \cos^{(\alpha-1)}(t/2) \sin(t/2) \right].$$

For $0 < t < \pi/\alpha$, $\sin(\alpha t/2) > \sin(t/2)$, and consequently $\nu'(t) > 0$. Thus, $\nu(t) > 0$ for $0 < t < \pi/\alpha$, and the desired inequality is established.

3. The coefficient problem. If $\phi(z) = A_1 z + A_2 z^2 + \dots$, $\psi(z) = B_1 z + B_2 z^2 + \dots$, and $A_1 = B_1$, then

$$h(z) = (\phi \circ \psi^{-1})(z) = z + \left(\frac{A_2 - B_2}{A_1^2} \right) z^2 + \dots$$

If we choose $\psi(z) = -\phi(-z)$, then for $\phi \in \Lambda_j$, $j = 1, 2$, we have $\psi \in \Lambda_j$, $h \in \sigma_j$, and $h(z) = z + (2A_2/A_1^2)z^2 + \dots$. Thus, to maximize the modulus of the second coefficient of h over σ_j it is sufficient to solve the extremal problem

$$\max_{\phi \in \Lambda_j} \left| \frac{A_2}{A_1^2} \right|. \quad (1)$$

The above choice of ψ relative to ϕ automatically yields an h in σ when $\phi \in \Lambda_1$ ($\sigma_1 \subset \sigma$), but when $\phi \in \Lambda_2$ this approach need not produce a univalent h . However, when $\psi(z) = -\phi(-z)$, $h^{-1}(z) = -h(-z)$, so if $h \in S$, then $h^{-1} \in S$ and $h \in \sigma$.

Netanyahu [3] showed that the solution of (1) with $j = 1$ is $2/3$ and that the extremal function maps $\mathbf{B}$ onto the plane less the slit $(-\infty, -1]$ and a slit along $|z| = 1$ centered at -1 . In Example 3 we generalize this function to obtain a $\phi \in \Lambda_2$ with $|A_2/A_1^2| > 2/3$. The function h so generated will be univalent for certain values of the parameters and hence, as discussed above, will be in σ .

EXAMPLE 3. Suppose $0 < \theta < \pi$, $2 - (\theta/\pi) < \alpha < 4$, and let $\mathbf{B}^+ = \{z \in \mathbf{B}: \operatorname{Im} z > 0\}$. Consider $\lambda(z) = z^\alpha$, $\lambda(1) = 1$, followed by $S(z) = e^{-i\theta}(z - e^{i\theta})/(z - e^{-i\theta})$. The first function sends $\mathbf{B}^+$ into $\mathbf{B}$, and the second one maps $\mathbf{B}$ onto the upper half-plane, sending radial segments $\{re^{it}: 0 < r < 1\}$ onto circular arcs from $e^{i\theta}$ to the real axis, meeting the real axis orthogonally. The arc $\{e^{it}: 0 < t < (2\pi - \theta)/\alpha\}$ is mapped by $S \circ \lambda$ onto $(-1, \infty)$, with $e^{i\theta/\alpha}$ going to 0. If g is a self-mapping of $\mathbf{B}^+$ which sends -1 to 1, 0 to $e^{i\theta/\alpha}$, and 1 to $e^{i(2\pi - \theta)/\alpha}$, then $S \circ \lambda \circ g$ maps $(-1, 1)$ to $(-1, \infty)$ and 0 to 0. Extending to $\mathbf{B}$ via the reflection principle gives the desired function ϕ . We note that $\phi \in \Lambda_1$ if $0 < \alpha < 2$ and $\phi \in \Lambda_2$ for $\alpha > 2$.

To obtain the function g , let $\mu(z) = [(1+z)/(1-z)]^2$ and let

$$T(z) = (az + b - a)/z,$$

where $a = \mu(e^{i(2\pi - \theta)/\alpha})$ and $b = \mu(e^{i\theta/\alpha})$. Then $g = \mu^{-1} \circ T \circ \mu$.

Let $\zeta = \phi^{-1}(e^{i\theta}) = \mu^{-1}((b-a)/(1-a))$. We shall consider only the case $\zeta = i$, i.e., $(b-a)/(1-a) = \mu(i) = -1$. This expresses θ implicitly as a continuous function of α in an interval about $\alpha = 2$ via the equation

$$m(\theta, \alpha) = \cot^2\left(\frac{\theta}{2\alpha}\right) - 2 \cot^2\left(\frac{2\pi - \theta}{2\alpha}\right) = 1. \quad (2)$$

Thus, we obtain one parameter families of functions g_α , T_α , ϕ_α . If $\alpha = 2$, then $\theta = 2 \arccos(1/3)$ and ϕ_2 is the extremal function for (1) with $j = 1$. Assume hereafter that α is larger than 2 but near 2.

FIGURE 1

Let $\hat{\delta}$ and $\hat{\gamma}$ denote the images of $[0, 1]$ and $[-1, 0]$, respectively, under $S \circ \lambda$. Then $\hat{\delta} = \{e^{it} : \theta \leq t < \pi\}$, and $\hat{\gamma}$ is a circular arc joining $e^{i\theta}$ and a point, $\hat{x}$, on the real axis. The curves $\hat{\delta}$, $\hat{\gamma}$, and $[-1, \hat{x}]$ bound the region, U , in the upper half-plane which is covered twice by $\phi_\alpha|B^+$. An examination of the mapping ϕ_α shows that $\phi_\alpha^{-1}(U)$ consists of two components as illustrated by the shaded regions in Figure 1. The curves γ , δ , η , in Figure 1 correspond to $\hat{\gamma}$, $\hat{\delta}$ and $\{e^{it} : 0 \leq t < \theta\}$, respectively, and $\phi_\alpha(x) = \hat{x}$. If V is the region bounded by γ , $[x, 1]$, and $\{e^{it} : 0 \leq t < \pi/2\}$, then $\phi_\alpha|V$ is one-one and $\phi_\alpha(V)$ is the upper half-plane less $\hat{\gamma}$. The restriction of ϕ_α to the region, W , bounded by $\{e^{it} : \pi/2 \leq t < \pi\}$, $[-1, y]$ and η is univalent, and $\phi_\alpha(W) = B^+$. Let $\psi_\alpha(z) = -\phi_\alpha(-z)$, $h_\alpha = \phi_\alpha \circ \psi_\alpha^{-1}$ and $\tilde{\eta}$ be the reflection of η in the imaginary axis. Now, ψ_α is one-one on $-\overline{W} = \{-\bar{z} : z \in W\}$, and $\psi_\alpha(-\overline{W}) = B^+$. Thus, to see that h_α is univalent, it is sufficient to show that $-\overline{W} \subset V$. We will do this by showing that for α sufficiently near 2 the curves γ and $\tilde{\eta}$ intersect only at i .

Since a and b are continuous functions of α , one can show that as α approaches 2, T_α^{-1} approaches T_2^{-1} uniformly relative to the chordal metric on the closed upper half-plane. It then follows that g_α^{-1} approaches g_2^{-1} uniformly on the closure of B^+ .

Let $\eta_2 = g_2^{-1}([0, i])$ and $\gamma_2 = g_2^{-1}([0, 1])$. By reflection g_2^{-1} is conformal at zero, so η_2 and γ_2 meet at i with angle $\pi/2$. There is a neighborhood $\mathcal{U}$ of zero and narrow sectors with vertices at zero, symmetric about $[0, i]$ and $[0, 1]$, respectively, such that the images of these sectors under $g_2^{-1}|_{\mathcal{U}}$ have positive angular separation at i . Since g_α^{-1} approaches g_2^{-1} uniformly on $\mathcal{U}$ as α approaches 2, there are choices of $\arg g_\alpha^{-1}$ and $\arg g_2^{-1}$ such that $\arg g_\alpha^{-1}$ approaches $\arg g_2^{-1}$ uniformly as α approaches

2. It follows that for α near 2 the images of the sectors under g_α^{-1} have positive angular separation at i . Thus, in some neighborhood of i , γ and $\tilde{\eta}$ cannot intersect except at i for all α sufficiently near 2. Outside this neighborhood, γ_2 and $\tilde{\eta}_2$ are a positive distance apart, and consequently so are γ and $\tilde{\eta}$ when α is sufficiently near 2.

Extensive but routine calculations show that for these functions ϕ_α ,

$$\frac{A_2}{A_1^2} = \frac{\sin \theta}{\alpha \sin(\theta/\alpha)} \left\{ 1 + \cos(\theta/\alpha) - \frac{1}{2} \left[\frac{\sin(\pi/\alpha)}{\sin((\pi - \theta)/\alpha)} + \frac{\sin((\pi - \theta)/\alpha)}{\sin(\pi/\alpha)} \right] \right\} - \cos \theta.$$

At the point $\alpha = 2$, $\theta = \theta_0 = 2 \arccos(1/3)$, the level set $m(\theta, \alpha) = 1$ given by (2) has slope $d\alpha/d\theta = 3/[3 \arccos(1/3) - \pi] \sim 5.44$. Thus

$$\vec{v} = (1, 3/[3 \arccos(1/3) - \pi])$$

is a vector tangent to the level curve (2) at $(\theta_0, 2)$ in the direction of increasing α . Further computation shows that the gradient of A_2/A_1^2 at $(\theta_0, 2)$ is $(.0112 \dots, 0)$, and consequently the directional derivative of A_2/A_1^2 in the direction $\vec{v}$ is $.002 \dots$. Thus, for α slightly larger than 2, A_2/A_1^2 takes values strictly larger than $2/3$. Using a programmable calculator with $\alpha = 2.1$ we obtain $\theta = 2.4785 \dots$ and $A_2/A_1^2 = .6674 \dots$.

It is an open question as to whether the maximum of $|a_2|$ over σ is as large as that over σ_2 . In regard to the latter problem, we show by Example 4 below that the solution of (1) with $j = 2$ is at least 0.7236.

EXAMPLE 4. Consider

$$f(z) = -z \frac{(1+b)z - 2b}{(1+b) - 2bz} \left[\frac{1-bz}{z-b} \right]^2, \quad 0 < b < 1.$$

We note that $f(\partial\mathbf{B}) \subset \partial\mathbf{B}$, since f is a product of functions with this property. Also, $f(\bar{z}) = \overline{f(z)}$. Calculations show

$$f'(z) = \frac{-2b^2(1+b)(1-bz)(1-z)^2[z^2 + (1-3b)z + 1]}{(1+b-2bz)^2(z-b)^3}.$$

We see that f has a pole of order 2 at b , a zero of order 2 at $1/b$, a critical point of order 2 at 1 and simple critical points at the roots, z_0 and $\bar{z}_0$, of $z^2 + (1-3b)z + 1 = 0$. These account for all the branch points of the image of f . Assume $\text{Im } z_0 > 0$ and note that $|z_0| = 1$. Taking these features into consideration we conclude that f maps $\mathbf{B}$ univalently onto the plane less the slit $[-1, \infty)$ and the slit along $|z| = 1$ from -1 to $f(z_0)$. Furthermore, f maps $[-1, b)$ onto $[-1, \infty)$. Let g be a self-mapping of $\mathbf{B}^+$ which sends -1 to -1 , 0 to 0, and 1 to b , and let $\phi = f \circ g$. Then $\phi(\mathbf{B}^+) = f(\mathbf{B}^+)$, and ϕ sends $(-1, 1)$ onto $(-1, \infty)$. Extending by the reflection principle to $\mathbf{B}$ we see that $\phi \in \Lambda_2$. The function g is given by $\mu^{-1} \circ T \circ \mu$, where $\mu(z) = [(1+z)/(1-z)]^2$ and $T(z) = \mu(b)z/[z + \mu(b) - 1]$.

After considerable calculation we obtain $A_2/A_1^2 = (-5b^3 + 3b + 2)/4$, which has a maximum value over $(0, 1)$ of $(1 + \sqrt{5})/2\sqrt{5} = 0.7236 \dots$, occurring at $b = 1/\sqrt{5}$.

This leads to the following natural questions. Is $|A_2/A_1^2| < (1 + \sqrt{5})/2\sqrt{5}$ for all $\phi \in \Lambda_2$? If so, then $|a_2| < (1 + \sqrt{5})/\sqrt{5} \sim 1.45$ for all $f \in \sigma_2$, and $|a_2|$ would likely be even less for $f \in \sigma$. In any case, the gap between $4/3^+$ and 1.51^- for $|a_2|$ when $f \in \sigma$ seems rather large.

REFERENCES

1. D. Bshouty, W. Hengartner and G. Schober, *Estimates for the Koebe constant and the second coefficient for some classes of univalent functions* (preprint).
2. M. Lewin, *On a coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc. **18** (1967), 63–68.
3. E. Netanyahu, *The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$* , Arch. Rational Mech. Anal. **32** (1969), 100–112.
4. T. J. Suffridge, *A coefficient problem for a class of univalent functions*, Michigan Math. J. **16** (1969), 33–42.

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