LIE GROUPS WHICH ADMIT FLAT LEFT INVARIANT METRICS

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ABSTRACT. Let G be a Lie group which admits a flat left invariant metric. We show that any nonflat left invariant metric on G has conjugate points and we describe how some of the conjugate points arise.

Introduction. Milnor [6] has shown that a Lie group G admits a flat left invariant metric if and only if the Lie algebra $\mathfrak{g}$ can be written an orthogonal direct sum $\mathfrak{h} \oplus \mathfrak{a}$ where $\mathfrak{h}$ is a commutative subalgebra, $\mathfrak{a}$ is a commutative ideal and ad_B is skew symmetric for every $B \in \mathfrak{h}$. The purpose of this paper is to prove the following.

THEOREM. *If G admits a flat left invariant metric, then any nonflat left invariant metric has conjugate points.*

We first recall that, if X and Y are left invariant vector fields, the Riemannian connection ∇ of a left invariant metric $\langle, \rangle$ satisfies

$$\nabla_X Y = \frac{1}{2} \{ \text{ad}_X Y - \text{ad}_X^* Y - \text{ad}_Y^* X \} \quad (1)$$

where $*$ denotes transpose with respect to the inner product $\langle, \rangle$ (see [1, p. 64]). Let e denote the identity element and g an arbitrary element of G . If $\bar{Y}$ is a right invariant vector field, then $\bar{Y}$ satisfies

$$\bar{Y}_g = L_{g*} [\text{Ad}(g^{-1}) \bar{Y}_e].$$

We will not distinguish between elements of the Lie algebra and their left translates and will therefore write

$$\bar{Y}_g = \text{Ad}(g^{-1}) \bar{Y}_e.$$

Let $E_1, \dots, E_n$ be an orthonormal basis for $\mathfrak{g}$, let $X \in \mathfrak{g}$ and let A denote the matrix of ad_X with respect to this basis. Suppose that $\bar{Y} = \sum_{i=1}^n f^i E_i$ and let $\underline{f}$ be $f^1, \dots, f^n$ written as a column vector. Along the one parameter group c generated by X we have

$$\bar{Y}_{c(t)} = \text{Ad}(\text{Exp}(-tX)) \bar{Y}_e$$

and so it follows by differentiation that, along c , $\underline{f}$ is a solution of the ordinary differential equation

$$\dot{\underline{f}} + A\underline{f} = 0. \quad (2)$$

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If X is perpendicular to the commutator $g' = [g, g]$, then $\text{ad}_Y^* X = 0$ for all $Y \in g$ and therefore it follows from (1) that the coefficient vector $\underline{f}$ of a parallel vector field $\sum_{i=1}^n f^i E_i$ along c is a solution of

$$\underline{\dot{f}} + \frac{1}{2}(A - A^*)\underline{f} = 0. \quad (3)$$

Before proceeding with the proof of the theorem we need the following lemma.

LEMMA. Let α^c be the complexification of α , then there exist $\text{ad}_{\mathfrak{b}}$ invariant subspaces $\alpha_1^c, \alpha_2^c, \dots, \alpha_l^c$ of α^c and real linear functionals $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_l$ on g such that

(i) $\alpha^c = \bigoplus_{j=1}^l \alpha_j^c$;

(ii) on each α_j^c , $\text{ad}_B = i\lambda_j(B)I_j$

for every $B \in \mathfrak{b}$, where I_j is the identity transformation on α_j^c .

PROOF. The mapping $B \rightarrow \text{ad}_B$ is a representation of the Abelian Lie algebra $\mathfrak{b}$ into $gl(\alpha^c)$. Let $\rho_1, \rho_2, \dots, \rho_l$ be the roots and $\alpha_1^c, \alpha_2^c, \dots, \alpha_l^c$ be the associated root spaces of the representation. Then $\alpha^c = \bigoplus_{j=1}^l \alpha_j^c$. For any root ρ and any $B \in \mathfrak{b}$, $\rho(B)$ is an eigenvalue of a skew symmetric linear transformation and, therefore, must have zero real part.

Furthermore, $(\text{ad}_B - \rho(B)I)^k U = 0$ for $U \in \alpha^c$ and $k > 1$ if and only if

$$(\text{ad}_B - \rho(B)I)U = 0.$$

It follows that each root ρ_j is of the form $\rho_j = i\lambda_j$ where λ_j is a real linear functional on $\mathfrak{b}$ and, on the corresponding root space, $\text{ad}_B = i\lambda_j(B)I_j$. Q.E.D.

PROOF OF THEOREM. Let $i\lambda$ be a root of $\text{ad}: \mathfrak{b} \rightarrow gl(\alpha^c)$ and let $U \in \alpha^c$ be a vector in the associated root space. Extend λ to a real linear functional on all of g by setting $\lambda = 0$ on α . Since α is an Abelian subalgebra it follows that for all $X \in g$

$$\text{ad}_X(U) = i\lambda(X)U.$$

If $\text{Exp}: g \rightarrow G$ denotes the Lie algebra exponential mapping, it follows that for $g = \text{Exp}(X)$

$$\text{Ad}(g)U = e^{i\lambda(X)}U.$$

Without loss of generality, G can be assumed to be connected, so we can write any $g \in G$ in the form

$$g = \text{Exp}(X_1)\text{Exp}(X_2) \cdots \text{Exp}(X_r)$$

for some $X_1, X_2, X_3, \dots, X_r \in g$, and so

$$\text{Ad}(g)U = \left\{ \prod_{j=1}^r e^{i\lambda(X_j)} \right\} U = e^{i\lambda(X)}U \quad (4)$$

where $X = \sum_{j=1}^r X_j$.

Suppose $U = Y_1 + iY_2$ and, for $i = 1, 2$, let $\bar{Y}_i$ be the right invariant vector field on G which coincides with Y_i at the identity element e . Since any right invariant vector field $\bar{Y}$ satisfies

$$\bar{Y}_g = \text{Ad}(g^{-1})\bar{Y}_e$$

it follows from (4) that, for any $g \in G$, we can find an element X of g such that

$$\bar{Y}_1(g) = \text{Re}\{e^{-i\lambda(X)}U\} \quad \text{and} \quad \bar{Y}_2(g) = \text{Im}\{e^{-i\lambda(X)}U\}.$$

Hence, for $i = 1, 2$, the length of $\bar{Y}_i$ is bounded above in any left invariant metric. If $\lambda \equiv 0$, $\bar{Y}_i$ has constant length. Otherwise $\exists Z \in \mathfrak{g}$ such that $\lambda(Z) \neq 0$ and then it follows from the periodicity of the sine and cosine functions that the length of $\bar{Y}_i$ attains an absolute maximum at some point along the one parameter subgroup generated by Z . By a result of O'Sullivan [7, Theorem B, p. 307] it follows that, if the length of a Killing vector field attains an absolute maximum, there are conjugate points unless the field is parallel. Hence there are conjugate points on G unless $\bar{Y}_1$ and $\bar{Y}_2$ are both parallel vector fields. But since $i\lambda$ was any root of the representation $\text{ad}: \mathfrak{b} \rightarrow \mathfrak{g}/(\alpha^\mathfrak{c})$ and since $\alpha^\mathfrak{c}$ is spanned by the root spaces, it follows that there are conjugate points on G unless all right invariant vector fields $\bar{Y}$ with $\bar{Y}_\mathfrak{e} \in \alpha$ are parallel.

Suppose all such vector fields are parallel in a left invariant metric $\langle, \rangle$. Let $X \in \alpha^\perp$, the orthogonal complement of α in $\mathfrak{g}$. Since α is an ideal and $\mathfrak{b}$ is Abelian, it follows that the commutator $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is contained in α and so X is perpendicular to $\mathfrak{g}'$. Therefore, if $Y \in \alpha$, it follows from (2) and (3) that

$$\text{ad}_X Y = \frac{1}{2}(\text{ad}_X Y - \text{ad}_X^* Y)$$

i.e.

$$\text{ad}_X Y = -\text{ad}_X^* Y. \quad (5)$$

Since α is an ideal, ad_X preserves α and so, by (5), ad_X^* must also preserve α . This implies that ad_X preserves $\alpha^\perp$, i.e.,

$$\text{ad}_X(\alpha^\perp) \subset \alpha^\perp.$$

Since $\mathfrak{g}' \subset \alpha$, we also have

$$\text{ad}_X(\alpha^\perp) \subset \alpha.$$

Therefore $\text{ad}_X(\alpha^\perp) = 0$ and it follows that $\alpha^\perp$ is an Abelian subalgebra.

We have now shown that $\mathfrak{g}$ splits as an orthogonal direct sum $\alpha \oplus \alpha^\perp$ where α is an Abelian ideal and $\alpha^\perp$ is an Abelian subalgebra. Further, by (5), ad_X is skew symmetric for every $X \in \alpha$. Hence, by Milnor [6], the metric $\langle, \rangle$ is flat. Q.E.D.

We will now give a partial description of the conjugate points arising in the nonflat metric. Using the notation of the proof, let $i\lambda$ be a root of $\text{ad}: \alpha^\perp \rightarrow \mathfrak{g}/(\alpha^\mathfrak{c})$ and $\alpha_\lambda^\mathfrak{c}$ be the root space for $i\lambda$. Extending λ to all of $\mathfrak{g}$ by setting $\lambda(\alpha) = 0$, we have for $X \in \mathfrak{g}$, $U = Y_1 + iY_2 \in \alpha_\lambda^\mathfrak{c}$,

$$\text{ad}_X U = i\lambda(X)U.$$

If $\lambda \equiv 0$ on $\mathfrak{g}$, then Y_1 and Y_2 are both in the center of $\mathfrak{g}$. So by [3, Theorem 1], for any $a, b \in \mathbb{R}$, $Y = aY_1 + bY_2$ is either parallel and, therefore, orthogonal to $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$, or the one parameter group $\text{Exp}(tY)$ is a geodesic with even order periodic sets of conjugate points.

If $\lambda \neq 0$, then any nonzero right invariant vector field $\bar{Y}$, defined by $\bar{Y}_\mathfrak{e} = aY_1 + bY_2$, $a, b \in \mathbb{R}$, is of bounded length and assumes maximal length at some point $g \in G$. Let $A_{\bar{Y}}$ be the $(1, 1)$ tensor field defined by

$$A_{\bar{Y}}X = -\nabla_X \bar{Y}$$

and let $f = \frac{1}{2}\langle \bar{Y}, \bar{Y} \rangle$.

Since $\bar{Y}$ is a Killing vector field, it follows that $A_{\bar{Y}}$ is skew symmetric, that $\text{grad } f = A_{\bar{Y}}\bar{Y}$ and that for any vector Z

$$R(Z, \bar{Y})\bar{Y} = -\nabla_Z(A_{\bar{Y}}\bar{Y}) - A_{\bar{Y}}^2(Z). \quad (6)$$

In particular,

$$\begin{aligned} \text{Ric}(\bar{Y}, \bar{Y}) &= -\text{div}(A_{\bar{Y}}\bar{Y}) - \text{Trace}(A_{\bar{Y}}^2) \\ &= -\Delta f - \text{Trace}(A_{\bar{Y}}^2) \end{aligned} \quad (7)$$

where Δ denotes the Laplacian. (See [5, p. 248 ff].)

Furthermore, the integral curve c of $\bar{Y}$ through g is a geodesic. Since f is constant and maximal along c , it follows from (6) and (7) that $\text{Ric}(\dot{c}, \dot{c}) > 0$ unless the restricted curvature tensor R_c , defined by $R_c(Z) = R(Z, \dot{c})\dot{c}$ for $Z \in T_c(G)$, vanishes. By [2], there are conjugate points on a geodesic with positive Ricci curvature. If $Y_0 = (L_{g^{-1}})_*\bar{Y}_g$, then $c_0 = L_{g^{-1}}c$, the one parameter group generated by Y_0 , is a geodesic with conjugate points unless R_{c_0} vanishes.

REMARK. Any compact manifold is finitely covered by the torus. Therefore if G has a discrete subgroup Γ such that the quotient G/Γ is compact, one can choose Γ so that the quotient is a torus. Any left invariant metric on G would then produce a Riemannian metric on the torus with the same curvature properties and the same Jacobi fields. It is a consequence of our theorem that any such metrics on the torus must have conjugate points if they are not flat. In [4], E. Hopf showed that all nonflat metrics on the two dimensional torus have conjugate points, but the question remains open in higher dimensions.

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