ON COUNTABLY GENERATED z -IDEALS OF $C(X)$ FOR FIRST COUNTABLE SPACES

ATTILIO LE DONNE

ABSTRACT. In paper [L], a question asked in [D] has been answered first by proving that: *if X is normal and first countable, then every countably generated z -ideal of $C(X)$ is pure*; then, by giving an example of a nonpure countably generated z -ideal of $C(X)$ in a σ -compact (hence normal) but not first countable space X .

In this paper a class $\mathcal{C}$ of topological spaces X whose $C(X)$ has a nonpure countably generated z -ideal is constructed; it is proved that $\mathcal{C}$ contains a space X which is first countable. So it is proved that in the proposition above the hypotheses "normal" and "first countable" are both essential.

Finally in §4 I prove, as announced in [L], that if X is a locally compact normal space, then every countably generated z -ideal of $C(X)$ is pure.

For the terminology and notations see [GJ], [D], [L].

1. The classes $\mathfrak{B}$ and $\mathcal{C}$. Let $\mathfrak{B}$ be the class of topological spaces B which are disjoint unions (but *not* topological sums) of three subspaces C, D, E , with the properties:

(P^I) $C = \omega \times D$ (ω is the discrete space of natural numbers);

(P^{II}) there exists $g: B \rightarrow R$ continuous such that $Z(g) = D$ and g is constant on the sets $\{n\} \times D \subset C$ for each $n \in \omega$;

(P^{III}) if $f: B \rightarrow R$ is continuous and vanishes for every $n \in \omega$ on a point $(n, d_n) \in C$ then f vanishes at some point of D ;

(P^{IV}) $D \subset \text{int}_B(B \setminus C)$.

Fix a bijection $u: \omega \rightarrow \omega \times \omega$, $u(s) = (s_1, s_2)$. For each $B \in \mathfrak{B}$ let

$$\Lambda(B) = B \times \omega \times \omega.$$

Let $\mu: \Lambda(B) \rightarrow \Lambda(B)$ be the (noncontinuous) map which is the identity on $(B \setminus D) \times \omega \times \omega$ and maps the point $x = (d, n, m) \in D \times \omega \times \omega$ into the point $\mu(x) = ((n_1, d), n_2, m + 1) \in C \times \omega \times \omega$. Clearly μ is idempotent (i.e., $\mu \circ \mu = \mu$).

We define $X(B)$ as the quotient of $\Lambda(B)$ by the equivalence relation which identifies x and $\mu(x)$, for every $x \in \Lambda(B)$.

The spaces $X(B)$ with B running over $\mathfrak{B}$ form the class $\mathcal{C}$.

We note that the class $\mathfrak{B}$ (and hence the class $\mathcal{C}$) is nonvoid: in fact, the "book" B of [L] satisfies (P^I)–(P^{IV}) with C the "top section", D the "back"; moreover $X(B)$ for such a "book" B is exactly the space X of [L].

2. Properties of the class $\mathcal{C}$. Let $X = X(B) \in \mathcal{C}$; put $\Sigma_n = B \times \omega \times \{n\}$. If $q: \Lambda(B) \rightarrow X(B)$ is the quotient map, put $Z_n = q(\bigcup_{k \geq n+1} \Sigma_k)$.

Received by the editors April 15, 1980 and, in revised form, July 14, 1980.
1980 *Mathematics Subject Classification*. Primary 54C40; Secondary 54G20.

© 1981 American Mathematical Society
0002-9939/81/0000-0274/\$01.75

LEMMA 2.1. Z_n is a zero set of X .

The proof is analogous to the construction of the sequence (f_n) of [L, 2.2], using the existence of the function g of property (P^{II}) .

LEMMA 2.2. $Z_{n+1} \subset \text{int}_X Z_n$.

PROOF. By property (P^{IV}) the set $D \times \omega \times \{n\}$ is contained in

$$\text{int}_\Lambda((B \setminus C) \times \omega \times \{n\}),$$

then

$$\left(\bigcup_{k \geq n+1} \Sigma_k \right) \cup (D \times \omega \times \{n\})$$

is contained in the interior of

$$\left(\bigcup_{k \geq n+1} \Sigma_k \right) \cup ((B \setminus C) \times \omega \times \{n\})$$

and these two sets are $q^{\leftarrow} \circ q$ stable.

LEMMA 2.3. Every continuous function $f \in C(X)$ vanishing on $X \setminus Z_1$ vanishes at some point of Z_n .

PROOF. The function $f \circ q: \Lambda \rightarrow \mathbb{R}$ vanishes at every point of $C \times \omega \times \{1\}$. Then by property (P^{III}) it vanishes at a point of $D \times \{k\} \times \{1\}$ for every $k \in \omega$. By the equivalence relation, $f \circ q$ vanishes at a point $((n, d_n^{(h)}, h, 2) \in C \times \{h\} \times \{2\}$ for every $h \in \omega$; then by (P^{III}) it vanishes at a point of $D \times \{h\} \times \{2\}$; and so on.

Then by Lemma 1.2 of [L], we have

PROPOSITION 2.1. If $X \in \mathcal{C}$ then $C(X)$ has a nonpure countably generated z -ideal.

3. Construction of a $B \in \mathfrak{B}$ such that $X(B)$ is a $T_{3\frac{1}{2}}$ (also zero-dimensional) first countable space. We put $D = \omega^\omega$, $C = \omega \times D$ and $E = \omega \times \omega \times D \times D$; and $B = (\omega \times \omega \times D \times D) \cup (\omega \times D) \cup D$ with the topology given by:

(I) the points of E are isolated;

(II) the points $x = (n, p) \in \omega \times D$ have as a countable neighbourhood base the sets

$$\{x\} \cup (\{h|h > k\} \times \{n\} \times D \times \{p\})$$

with k running over ω ;

(III) the points $p = (m_h)_{h \in \omega} \in D$ have as a countable neighbourhood base the sets

$$\{p\} \cup (\{(m_h, h)|h > k\} \times \{p\} \times D)$$

with k running over ω .

It may be verified that each countable neighbourhood base, as given, consists of clopen sets; hence B is a T_2 zero-dimensional first countable space.

LEMMA 3.1. $B \in \mathfrak{B}$.

PROOF. Properties (P^I) and (P^{IV}) are obvious.

Property (P^{II}) . Define $\phi: B \rightarrow \mathbb{R}$ to be zero on D and put $\phi(m, n, p, q) = 2^{-n}$ and $\phi(n, p) = 2^{-n}$.

Property (P^{III}). Let $f: B \rightarrow R$ be continuous and let $p_n \in D$ such that $f(n, p_n) = 0$. Then there exists k_n such that on the set $\{k_n\} \times \{n\} \times D \times \{p_n\}$ the function $|f|$ is smaller than $1/n$. Now if $p = (k_n)_n$ we have $f(p) = 0$.

As we have said in §2, $X(B)$ consists of the equivalence classes $\{n, \mu(x)\}$ of $\Lambda(B)$ with x running over $(E \cup D) \times \omega \times \omega$.

If $x \in E \times \omega \times \omega$, we have $x = \mu(x)$ and the point x is isolated in $\Lambda(B)$ and so in $X(B)$. If $x \in D \times \omega \times \omega$ we can take the union of a neighbourhood of type (III) of x with one of type (II) of $\mu(x)$; this union is clopen in $\Lambda(B)$ and $q^\leftarrow \circ q$ stable: then also $q(x)$ has clopen neighbourhood base in $X(B)$. If $x \in C \times \omega \times \{1\}$, we have $\mu(x) = x$ and it suffices to take the neighbourhood base of x of type (III).

This proves that $X(B)$ is a first countable Hausdorff zero-dimensional space.

4. Locally compact normal space.

LEMMA 4.1. *If X is locally compact, then every countably generated z -ideal I of $C(X)$ has a sequence of generators $f_0, f_1, \dots$ such that $\text{int}_X Z(f_n) \supset Z(f_{n+1})$ for every $n \in N$.*

The proof is analogous to that of Lemma 1.5 of [L]. Let $g_0, g_1, \dots$ be a sequence of generators of I with $(g_0) \subset (g_1) \subset \dots$ and $g_0 < g_1 < \dots$. Assume that $\text{int}_X Z(g_0)$ fails to contain any $Z(g_n)$; then for each n there exists a $p_n \in \text{cl}_X \text{coz}(g_0) \cap Z(g_n)$. Let V_n be a compact neighbourhood of p_n in X . There is a sequence $x_m^n \in V_n$ with $0 \neq |g_0(x_m^n)| < |g_n(x_m^n)| < 1/m$. $B_n = \text{cl}_X \{x_m^n: m \in N\}$ is compact. Besides $B_n \setminus \{x_m^n: m \in N\}$ is not void and is contained in $Z(g_n)$. Now $S_n = \text{cl}_{\beta X} Z(g_0) \cup B_n$ is a closed subset of βX ; the function which is zero on $\text{cl}_{\beta X} Z(g_0)$ and is $|g_n(x_m^n)|^{1/2}$ on $\{x_m^n: m \in N\}$ is continuous on S_n ; hence it has a continuous extension $h_n^* \in C(\beta X)$. Let $h_n = h_n^*|_X$. Put $h = \sum_{n=0}^\infty (h_n \wedge 2^{-n})$; by uniform convergence $h \in C(X)$. Clearly $Z(h) \supset Z(g_0)$; but h cannot be a multiple (in $C(X)$) of any g_n . For if $h = ug_k$, as in [L, Lemma 1.5] we see that u cannot be continuous at any point of $B_n \setminus \{x_m^n: m \in N\}$.

Then by [L, 1.3(ii)] we have

PROPOSITION 4.2. *If X is normal and locally compact, then every countably generated z -ideal of $C(X)$ is pure.*

REFERENCES

- [D] G. De Marco, *On the countably generated z -ideals of $C(X)$* , Proc. Amer. Math. Soc. **31** (1972), 574–576.
- [GJ] L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, New York, 1960.
- [L] A. Le Donne, *On a question concerning countably generated z -ideals of $C(X)$* , Proc. Amer. Math. Soc. **80** (1980), 505–510.

SEMINARIO MATEMATICO, UNIVERSITÀ DI PADOVA, VIA BELZONI 7, 35100 PADOVA, ITALY