ON THE DIMENSION OF CUBIC μ -SPACES

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ABSTRACT. Let X be the countable product of special σ -metric spaces defined below. Then it is proved that $\text{Ind } X < n$ if and only if there exists a σ -closure-preserving open base $\mathcal{U}$ for X such that $\text{Ind } B(W) < n - 1$ for every $W \in \mathcal{U}$.

Let (A) be the characterization of large inductive dimension $\text{Ind } X$ of a topological space X as follows:

(A) $\text{Ind } X < n - 1$ if and only if there exists a σ -closure-preserving open base $\mathcal{U}$ for X such that $\text{Ind } B(W) < n - 1$ for every $W \in \mathcal{U}$.

J. Nigata in [7] proposed the problem whether (A) holds or not for every M_1 -space X . The author in [2, Theorem 1] proved that (A) holds for every paracompact σ -metric space X with a special scale. In this paper, it is proved that (A) holds for the countable product X of paracompact σ -metric spaces with special scales, and this generalizes the above result.

In the sequel, all maps are assumed to be continuous and onto and all spaces to be Hausdorff. N always denotes the positive integers.

DEFINITION 1 (K. NAGAMI [3]). A space X is called a σ -metric space if $X = \bigcup \{X_i : i \in N\}$, where each X_i is a closed metrizable subset of X . Such $\{X_i\}$ is called a *scale* of X . A scale $\{X_i\}$ is called *monotone* if $X_i \subset X_{i+1}$ for every $i \in N$.

The following argument is due to [3]: If X is a paracompact σ -metric space with $\text{Ind } X < n$ and with a monotone scale $\{X_i\}$, then there exists a contraction ρ of X onto a metric space $\hat{X}$ with $\text{Ind } \hat{X} < n$ such that for each $i \in N$ $\rho|X_i$ is a homeomorphism of X_i onto a closed subspace $\rho(X_i)$ of $\hat{X}$. Such a pair $(\hat{X}, \rho)$ is called a *replica* of X .

DEFINITION 2 (K. NAGAMI [5]). Let F be a closed set of a space X . An open cover $\mathcal{U}$ of $X - F$ is called an *anticover* of F . $\mathcal{U}$ is called to be *uniformly approaching* (to F) if for every open set G of X , $\overline{S(X - G, \mathcal{U})} \cap F \cap G = \emptyset$.

Every closed set of a metrizable space has a uniformly approaching anticover [5, Remark 4.5].

DEFINITION 3. A space X is called to have a *special scale* $\{X_i : i \in N\}$ if $\{X_i\}$ is a scale of X such that each X_i has a uniformly approaching anticover. A space X is called a *special σ -metric* if X is a paracompact σ -metric space with a special scale.

By a routine check it is seen that a space X has a special scale if and only if X has a special and monotone one. Therefore in the discussion below we do not

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distinguish between usual and monotone scales. The author in [2] proved the validity of (A) for a special σ -metric space X and for a space X embedded in the countable product of L -spaces, which are defined by Nagami [5] to be paracompact σ -spaces such that every closed set has an approaching anticover, where an anticover $\mathcal{U}$ of F is called *approaching* if, for every open neighborhood U of F , $\overline{S(X - U, \mathcal{U})} \cap F = \emptyset$. We shall give examples in order to consider the situation of spaces treated in this paper.

EXAMPLE 1. *There exists a special σ -metric space which is not an L -space. Let $X = \bigcup \{X(n): n \in N\}$, where every $X(n)$ is a triangle with the usual topology which intersects with only $I = [0, 1]$ (see the figure). Let X have the weak topology with respect to $\{X(n): n \in N\}$. Since each closed set of a metric space has a uniformly approaching anticover, X is easily seen to be a special σ -metric space with a special scale $\{X(n): n \in N\}$. But X is not an L -space because the closed set $F = \{1\}$ of X does not have an approaching anticover. To see this, let $\mathcal{V}$ be an arbitrary anticover of F in X . Let $\{p_n\} \subset I$ be a sequence of points which converges to F in I . There exists for each n a $V_n \in \mathcal{V}$ with $p_n \in V_n$. Take an open neighborhood $W(n)$ of I in $X(n)$ such that $(X(n) - W(n)) \cap V_n \neq \emptyset$, as indicated in the figure. Then $W = \bigcup \{W(n): n \in N\}$ is an open neighborhood of I , and hence of F in X such that $V_n \cap (X - W) \neq \emptyset$ for every n . Therefore $\overline{S(X - W, \mathcal{V})} \cap F \neq \emptyset$, proving that $\mathcal{V}$ is not approaching.*

EXAMPLE 2. *The countable product of special σ -metric spaces need not be a special σ -metric space.*

The space X in the above example is not metrizable. Therefore by the result of Nagami [3, Theorem 1] that the countable product of nonmetrizable spaces is not σ -metric, the countable product X^{ω_0} is not σ -metric.

EXAMPLE 3. *There exists the countable product X of L -spaces such that X is not a special σ -metric space.*

Consider the subspace $T = \bigcup \{I(n): n \in N\}$ of X described in Example 1, where $I(n) \subset X(n)$ and $I(n) \cap I(m) = \{1\}$ for every $n, m \in N, n \neq m$ (see the figure). Then T is an L -space and not metrizable. The countable product T^{ω_0} is not a σ -metric space by the same argument as in Example 2.

EXAMPLE 4. *There exists a cubic μ -space (the countable product of paracompact σ -metric spaces [4]) which is neither (i) the countable product of special σ -metric spaces nor (ii) embeddable in the countable product of L -spaces.*

Let X be Heath's butterfly space which is shown by him to be a paracompact σ -metric space but not a stratifiable space. Since, as shown essentially in the proof of [2, Theorems 1, 2], the space of type (i) or (ii) is M_1 , X cannot be either of them.

The author does not know the difference between spaces of type (i) and (ii) in the above example.

LEMMA 1. *Let $X \subset X_1 \times X_2 \times X_3$, where X_1, X_2 are special σ -metric spaces and X_3 a metric space. Suppose $\text{Ind } X \leq n$. Then there exists a closed map f of a paracompact σ -metric space Z with $\text{Ind } Z \leq 0$ onto X satisfying the following:*

(1) $\text{ord } f \leq n + 1$, and

(2) *if $Z' \subset Z$, then there exists a σ -closure-preserving collection $\mathcal{V}$ of clopen (closed and open) sets of Z' such that if*

$$p \in (f|Z')^{-1}((P_1 \times P_2 \times X_3) \cap X)$$

for a pair of open sets P_1, P_2 of X_1, X_2 , respectively, then

$$p \in V \subset (f|Z')^{-1}((P_1 \times P_2 \times X_3) \cap X)$$

for some $V \in \mathcal{V}$.

PROOF. Let $\{X_{1m}\}, \{X_{2m}\}$ be special scales of X_1, X_2 , and $(X'_1, \rho_1), (X'_2, \rho_2)$ be replicas of X_1, X_2 , respectively. Let

$$\rho = \rho_1 \times \rho_2 \times \text{id}_{X_3}|X: X \rightarrow \hat{X} = \rho(X) \subset X'_1 \times X'_2 \times X_3,$$

$$X_m = X \cap (X_{1m} \times X_{2m} \times X_3).$$

Then $(\hat{X}, \rho)$ is a replica of X such that $\rho|X_m$ is a homeomorphism of X_m onto a closed subspace $\rho(X_m)$ of $\hat{X}$. Thus $\text{Ind } \hat{X} \leq n$ follows easily. Let g be a closed map of a metric space Y onto $\hat{X}$ such that $\text{Ind } Y \leq 0$ and $\text{ord } g \leq n + 1$. Construct

$$Z = \{(x, y) \in X \times Y: \rho(x) = g(y)\},$$

$$Z_m = (X_m \times Y) \cap Z, \quad m \in N.$$

Let f, σ be the projections of Z onto X, Y , respectively. Then $\sigma|Z_m: Z_m \rightarrow \sigma(Z_m)$ is a homeomorphism of a metric space Z_m onto a closed subspace $\sigma(Z_m)$ of Y . It was shown in [3] that f is a closed map of the paracompact σ -metric space Z with $\text{Ind } Z \leq 0$ onto X such that $\text{ord } f \leq n + 1$. Thus we shall show that Z has the property (2). It suffices to prove (2) for the case $Z' = Z$. Let m be fixed for a while. Let $\mathcal{W}_1 = \{W_{1\alpha}: \alpha \in A\}, \mathcal{W}_2 = \{W_{2\beta}: \beta \in B\}$ be uniformly approaching anti-covers of X_{1m}, X_{2m} , which are assumed to be locally finite in $X_1 - X_{1m}, X_2 - X_{2m}$, respectively. Let

$$\mathcal{F}_1 = \{F_{1\alpha}: \alpha \in A\}, \quad \mathcal{F}_2 = \{F_{2\beta}: \beta \in B\}$$

be relatively closed covers of $X_1 - X_{1m}, X_2 - X_{2m}$, respectively, such that $F_{1\alpha} \subset W_{1\alpha}, F_{2\beta} \subset W_{2\beta}$ for each pair of $\alpha \in A, \beta \in B$. Put

$$D_1 = (((X_1 - X_{1m}) \times X_2 \times X_3) \cap X) \times Y) \cap Z,$$

$$D_2 = (((X_1 \times (X_2 - X_{2m}) \times X_3) \cap X) \times Y) \cap Z.$$

By the subset theorem $\text{Ind } D_1 < 0$, $\text{Ind } D_2 < 0$. For each pair of $\alpha \in A$, $\beta \in B$ there exist clopen sets $O_1(\alpha)$, $O_2(\beta)$ in D_1 , D_2 , respectively, such that

$$\begin{aligned} (((F_{1\alpha} \times X_2 \times X_3) \cap X) \times Y) \cap Z &\subset O_1(\alpha) \\ &\subset (((W_{1\alpha} \times X_2 \times X_3) \cap X) \times Y) \cap Z, \end{aligned} \quad (1)$$

$$\begin{aligned} (((X_1 \times F_{2\beta} \times X_3) \cap X) \times Y) \cap Z &\subset O_2(\beta) \\ &\subset (((X_1 \times W_{2\beta} \times X_3) \cap X) \times Y) \cap Z. \end{aligned} \quad (2)$$

For each subset $\delta \subset A$ we put

$$V(\delta) = [(((X_{1m} \times X_2 \times X_3) \cap X) \times Y) \cap Z] \cup [\cup \{O_1(\alpha): \alpha \in \delta\}].$$

Similarly for each $\delta \subset B$ we put

$$W(\delta) = [(((X_1 \times X_{2m} \times X_3) \cap X) \times Y) \cap Z] \cup [\cup \{O_2(\beta): \beta \in \delta\}].$$

Since X_{1m} is a closed metrizable subspace, there exist collections $\mathcal{U}(m) = \cup \{\mathcal{U}_k(m): k \in N\}$, $\mathcal{F}(m) = \cup \{\mathcal{F}_k(m): k \in N\}$ of X_1 such that

(3) $\mathcal{U}_k(m)$ is a locally finite open collection of X_1 ,

(4) $\mathcal{F}_k(m) = \{F(U): U \in \mathcal{U}_k(m)\}$ is a cover of X_{1m} by closed sets of X_{1m} such that $F(U) \subset U$ for every $U \in \mathcal{U}_k(m)$, and

(5) $\mathcal{U}(m)|_{X_{1m}}$ is a base for the subspace X_{1m} in the sense that if $p \in G$ for an open set G of X_{1m} , then there exists $U \in \mathcal{U}(m)$ such that $p \in F(U) \subset U \cap X_{1m} \subset G$. Similarly let

$$\mathcal{V}(m) = \cup \{\mathcal{V}_i(m): i \in N\}, \quad \mathcal{H}(m) = \cup \{\mathcal{H}_i(m): i \in N\}$$

be collections of X_2 such that

(6) $\mathcal{V}_i(m)$ is a locally finite open collection of X_2 ,

(7) $\mathcal{H}_i(m) = \{H(V): V \in \mathcal{V}_i(m)\}$ is a cover of X_{2m} by closed sets of X_{2m} such that $H(V) \subset V$ for every $V \in \mathcal{V}_i(m)$, and

(8) $\mathcal{V}(m)|_{X_{2m}}$ is a base for the subspace X_{2m} in the sense that if $p \in G$ for an open set G of X_{2m} , then there exists $V \in \mathcal{V}(m)$ such that $p \in H(V) \subset V \cap X_{2m} \subset G$.

Take $k, i \in N$, $U \in \mathcal{U}_k(m)$ and $V \in \mathcal{V}_i(m)$. Since $\text{Ind } Z < 0$, there exists a clopen set $W(U, V)$ of Z such that

$$\begin{aligned} (((F(U) \times H(V) \times X_3) \cap X) \times Y) \cap Z &\subset W(U, V) \\ &\subset (((U \times V \times X_3) \cap X) \times Y) \cap Z. \end{aligned}$$

For each pair of $U \in \mathcal{U}_k(m)$, $V \in \mathcal{V}_i(m)$ let $\Delta(U, V)$ be the collection of all $\lambda = (\delta_1, \delta_2)$ with $\delta_1 \subset A$ and $\delta_2 \subset B$ such that $W(U, V) \cap V(\delta_1)$, $W(U, V) \cap W(\delta_2)$ are open in Z . Put

$$\mathcal{V}(U, V) = \{V_\lambda = W(U, V) \cap V(\delta_1) \cap W(\delta_2): \lambda = (\delta_1, \delta_2) \in \Delta(U, V)\},$$

$$\mathcal{V}_{ki}(m) = \cup \{\mathcal{V}(U, V): U \in \mathcal{U}_k(m), V \in \mathcal{V}_i(m)\},$$

$$\mathcal{V}_m = \cup \{\mathcal{V}_{ki}(m): k, i \in N\},$$

$$\mathcal{V} = \cup \{\mathcal{V}_m: m \in N\}.$$

Then $\mathcal{V}$ satisfies the following:

Claim 1. $\mathcal{V}$ is a σ -closure-preserving collection of Z .

To see this, it suffices to prove that $\mathcal{V}(U, V)$ is closure-preserving for each pair $U \in \mathcal{U}_k(m)$, $V \in \mathcal{V}_i(m)$ because every point of Z has a neighborhood which meets $W(U, V)$ for at most finitely many U, V by (3) and (6) and because $V_\lambda \subset W(U, V)$ for every $\lambda \in \Delta(U, V)$. Let Δ_0 be an arbitrary subset of $\Delta(U, V)$ and suppose $p \in \overline{\bigcup \{V_\lambda : \lambda \in \Delta_0\}}$. Observe

$$\overline{\bigcup \{V_\lambda : \lambda \in \Delta_0\}} \cap Z_m \subset W(U, V) \cap Z_m. \quad (9)$$

Recall that $\{O_1(\alpha) : \alpha \in A\}$, $\{O_2(\beta) : \beta \in B\}$ are locally finite in D_1, D_2 , respectively, by (1) and (2). We consider the following cases to show $p \in \bar{V}_\lambda$ for some $\lambda \in \Delta_0$.

Case 1. If $p \in (((X_1 - X_{1m}) \times (X_2 - X_{2m}) \times X_3) \cap X) \times Y \cap Z$, then $p \in \bar{V}_\lambda$ for some $\lambda \in \Delta_0$ because $\{O_1(\alpha) \cap O_2(\beta) \cap W(U, V) : \alpha \in A, \beta \in B\}$ is locally finite at p .

Case 2. If $p \in Z_m$, then $p \in \bar{V}_\lambda$ for every $\lambda \in \Delta_0$ by (9).

Case 3. Suppose $p \in (((X_1 - X_{1m}) \times X_{2m} \times X_3) \cap X) \times Y \cap Z$ and suppose $p \notin \bar{V}_\lambda$ for every $\lambda \in \Delta_0$. $p \notin \bar{V}_\lambda$ implies $p \notin \overline{V(\delta_1) \cap W(U, V)}$ for $\overline{V(\delta_1) \cap W(U, V)} \subset \bar{V}_\lambda \cup (W(U, V) - W(\delta_2))$. Since $\{V(\delta_1) \cap W(U, V) : \delta_1 \subset A\}$ is closure-preserving in the subspace D_1 , there exists an open neighborhood $O(p)$ of p such that

$$O(p) \cap [\bigcup \{V(\delta_1) \cap W(U, V) : \lambda = (\delta_1, \delta_2) \in \Delta_0\}] = \emptyset.$$

Consequently we have $O(p) \cap [\bigcup \{V_\lambda : \lambda \in \Delta_0\}] = \emptyset$. This is a contradiction, and therefore $p \in \bar{V}_\lambda$ for some $\lambda \in \Delta_0$. This argument applies to the last case $p \in (((X_{1m} \times (X_2 - X_{2m}) \times X_3) \cap X) \times Y) \cap Z$.

Claim 2. $\mathcal{V}$ is a collection of clopen sets of Z .

Take $V_\lambda \in \mathcal{V}_m$. Then by an argument similar to the above one, we can show $B(V_\lambda) = \emptyset$.

Claim 3. If $p \in f^{-1}((P_1 \times P_2 \times X_3) \cap X)$ for a pair of open sets P_1, P_2 of X_1, X_2 , respectively, then $p \in V_\lambda \subset f^{-1}((P_1 \times P_2 \times X_3) \cap X)$ for some $V_\lambda \in \mathcal{V}$.

Since $Z = \bigcup \{Z_m : m \in N\}$, $p \in Z_m$ for some $m \in N$. By (5) and (8) there exist $U \in \mathcal{U}_k(m)$, $V \in \mathcal{V}_i(m)$ such that

$$\begin{aligned} p \in f^{-1}((F(U) \times H(V) \times X_3) \cap X) &\subset f^{-1}((U \times V \times X_3) \cap X) \cap Z_m \\ &\subset f^{-1}((P_1 \times P_2 \times X_3) \cap X). \end{aligned}$$

Since $\mathcal{W}_1, \mathcal{W}_2$ are uniformly approaching, there exist $\delta_1 \subset A$ and $\delta_2 \subset B$ such that

$$V_1 = f^{-1}(((P_1 \cap X_{1m}) \times X_2 \times X_3) \cap X) \cup [\bigcup \{O_1(\alpha) : \alpha \in \delta_1\}],$$

$$V_2 = f^{-1}((X_1 \times (P_2 \cap X_{2m}) \times X_3) \cap X) \cup [\bigcup \{O_2(\beta) : \beta \in \delta_2\}]$$

are open sets of Z such that

$$V_1 \subset f^{-1}((P_1 \times X_2 \times X_3) \cap X), \quad V_2 \subset f^{-1}((X_1 \times P_2 \times X_3) \cap X).$$

Thus we have

$$\begin{aligned} p \in W(U, V) \cap V_1 \cap V_2 &= W(U, V) \cap V(\delta_1) \cap W(\delta_2) = V_\lambda \\ &\subset f^{-1}(((P_1 \times P_2 \times X_3) \cap X)), \quad \lambda \in \Delta(U, V). \end{aligned}$$

This completes the proof.

COROLLARY. Let X_i be special σ -metric spaces for $i = 1, \dots, m-1$, X_m a metric space and $X \subset \prod_{i=1}^m X_i$. Let (X'_i, ρ_i) be the replica of X_i and put

$$\rho = \rho_1 \times \dots \times \rho_{m-1} \times \rho_{m-1} \times \text{id}_{X_m} | X: X \rightarrow \rho(X) = P_1.$$

Let f_1 be a closed map of a metric space Q_1 with $\text{Ind } Q_1 < 0$ onto P_1 such that $\text{ord } f_1 \leq n+1$. Then there exists a closed map f_m of a paracompact σ -metric space Q_m with $\text{Ind } Q_m < 0$ onto X such that

- (1) $\text{ord } f_m \leq n+1$,
- (2) if $Q_0 \subset Q_m$, then there exists a σ -closure-preserving collection $\mathcal{W}$ of clopen sets of Q_0 such that if

$$p \in (f_m|Q_0)^{-1}((O_1 \times \dots \times O_{m-1} \times X_m) \cap X)$$

for open sets $O_1, \dots, O_{m-1}$ of $X_1, \dots, X_{m-1}$, respectively, then

$$p \in W \subset (f_m|Q_0)^{-1}((O_1 \times \dots \times O_{m-1} \times X_m) \cap X)$$

for some $W \in \mathcal{W}$; and

- (3) there exists a contraction h of Q_m onto Q_1 such that $f_1 h = \rho f_m$ and (Q_1, h) is a replica of Q_m .

In the proof of the next theorem, the notion of irreducible maps is used. A continuous surjection $f: X \rightarrow Y$ is called an *irreducible map* if $f(A) \neq Y$ for every proper closed set A of X .

THEOREM. Let $X = \prod X_i$ be the countable product of special σ -metric spaces X_i . Then X satisfies (A).

PROOF. By [2, Lemma 3] each X_i is an M_1 -space, and consequently X is an M_1 -space. Therefore the if part of (A) follows from [7, Lemma 7]. The only if part of (A): Let X'_i be a replica of X_i . For each $i \in N$ put

$$P_i = \prod_{j < i} X_j \times \prod_{j > i} X'_j.$$

Then P_i is a paracompact σ -metric space and P_1 is a metric space. Each P_i has a replica P_1 for $i \geq 2$. As seen in the proof of [4, Theorem 2], $X = \varprojlim \{P_i, g_j^i\}$, where g_j^i with $i > j$ is a natural projection and P_i onto P_j , and $\text{Ind } P_i < \aleph$ for each $i \in N$. Let f_1 be a closed map of a zero-dimensional metric space Q_1 onto P_1 such that $\text{ord } f_1 \leq n+1$. For each $i \geq 2$, by Corollary 1 there exist a closed map f_i of a zero-dimensional paracompact σ -metric space Q_i onto P_i such that

- (1) $\text{ord } f_i \leq n+1$,
- (2) if $Q'_i \subset Q_i$, then there exists a σ -closure-preserving collection $\mathcal{V}_i$ of clopen sets of Q'_i such that if

$$p \in (f_i|Q'_i)^{-1}\left(O_1 \times \dots \times O_{i-1} \times \prod_{j > i} X'_j\right)$$

for open sets $O_1, \dots, O_{i-1}$ of $X_1, \dots, X_{i-1}$, respectively, then

$$p \in V \subset (f_i|Q_i')^{-1}\left(O_1 \times \dots \times O_{i-1} \times \prod_{j>i} X_j'\right)$$

for some $V \in \mathcal{V}_i$, and a contraction $h_i^i: Q_i \rightarrow Q_1$ such that $f_1 h_1^i = g_1^i f_i$. Each $h_j^i: Q_i \rightarrow Q_j$ for $i > j$ is defined by $h_j^i = (h_1^i)^{-1} h_1^j$. Since Q_j has the base $\{(h_1^j)^{-1}(U) \cap f_j^{-1}(V): U, V \text{ are open in } Q_1, P_j, \text{ respectively}\}$, h_j^i is continuous. Let Z be the inverse limit of $\{Q_i, h_j^i\}$ and define a map $f: Z \rightarrow X$ by $f(z) = \langle f_i(z_i) \rangle$ for $z = \langle z_i \rangle \in Z$. As shown in the proof of [4, Theorem 2], f is a closed map with $\text{ord } f \leq n+1$ and $\dim Z \leq 0$. Since f is a perfect map, by [1, Lemma 3.2] there exists a closed subspace Z_0 of Z such that $f_0 = f|Z_0$ is an irreducible map of Z_0 onto X . Let $h_i: Z \rightarrow Q_i$ be the projection for each $i \in N$. By (2) there exists a σ -closure-preserving collection $\mathcal{V}_i$ of clopen sets of $h_i(Z_0)$ satisfying the condition (2) with $Q_i' = h_i(Z_0)$. Then

$$\mathcal{W} = \bigcup \{(h_i|Z_0)^{-1}(\mathcal{V}_i): i \in N\}$$

is a σ -closure-preserving collection of clopen sets of Z_0 . We shall show the following: If $p \in f_0^{-1}(U)$ for an open set U of X , then $p \in W \subset f_0^{-1}(U)$ for some $W \in \mathcal{W}$. We can take an $i \in N$ and an open set U_i of the form $U_i = V_1 \times \dots \times V_{i-1} \times \prod_{j>i} X_j'$ with each V_i open in X_i such that $p \in f_0^{-1}(g_i^{-1}(U_i)) \subset f_0^{-1}(U)$, where $g_i: X \rightarrow P_i$ is the projection. Then

$$h_i(p) \in (f_i|Z_0)^{-1}(U_i).$$

By (2) there exists $V \in \mathcal{V}_i$ such that $h_i(p) \in V \subset (f_i|Z_0)^{-1}(U_i)$. This implies

$$p \in (h_i|Z_0)^{-1}(V) \subset f_0^{-1}(g_i^{-1}(U_i))$$

and thus we have $p \in W \subset f_0^{-1}(U)$ for some $W \in \mathcal{W}$. Write

$$\mathcal{W} = \bigcup \{\mathcal{W}_i: i \in N\},$$

where each $\mathcal{W}_i$ is closure-preserving in Z_0 . Without loss of generality we can assume that $\mathcal{W}_i \subset \mathcal{W}_{i+1}$ and each $\mathcal{W}_i$ is closed under finite unions. Put

$$\mathcal{H}_i = \{H(W) = X - f_0(Z_0 - W): W \in \mathcal{W}_i\}, \quad \mathcal{H} = \bigcup \{\mathcal{H}_i: i \in N\}.$$

Then it follows from [1, Theorem 3.3] that $\mathcal{H}$ is a σ -closure-preserving base for X . It remains to prove that $\text{Ind } B(H(W)) \leq n-1$ for every $W \in \mathcal{W}$. Set

$$T = f_0^{-1}(B(H(W))) \cap (Z_0 - W).$$

Then $g_0 = f_0|T$ is a closed map of a normal space T onto $B(H(W))$ such that $\text{ord } g_0 \leq n$. Thus by [6, Corollary 22-12], $\text{Ind } B(H(W)) \leq n-1$. This completes the proof.

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