A NOTE ON THE THOM ISOMORPHISM

MARK MAHOWALD¹ AND NIGEL RAY

ABSTRACT. The generalized homology version of the Thom isomorphism theorem is exploited to give easy proofs of several recent theorems.

Let Ω be an H -space with homotopy inverse $x \rightarrow x^{-1}$, and let $f: \Omega \rightarrow BF$ classify a stable spherical fibration over Ω . The following result is proved in [2].

THEOREM 1. *If f is an H -map, then the Thom spectrum $T(f)$ is a ring spectrum, and there is a homotopy equivalence $\alpha: T(f) \wedge \Omega_+ \xrightarrow{\cong} T(f) \wedge T(f)$ (where $+$ signifies a disjoint base point).*

For many applications, it is important that α is given by the Thomification of an explicit map $g: \Omega \times \Omega \rightarrow \Omega \times \Omega$, where $g(x, y) = (xy^{-1}, y)$. For details, see [2].

Our purpose here is to note that Theorem 1 follows from a strong form of the Thom isomorphism theorem in generalized homology. Specifically, we assume given a stable spherical fibration $\nu: X \rightarrow BF$ over a locally finite CW complex X , and a ring spectrum E orienting ν ; i.e. a Thom class $U: T(\nu) \rightarrow E$ whose restriction to a fibre $S^0 \hookrightarrow T(\nu)$ is the unit of E . Then we have

THEOREM 2. *There is a homotopy equivalence $\alpha(U): E \wedge T(\nu) \rightarrow E \wedge X_+$ which on homotopy groups induces the traditional Thom isomorphism*

$$\phi_U: E_*(T(\nu)) = \pi_*(E \wedge T(\nu)) \xrightarrow[\alpha(U)_*]{\cong} \pi_*(E \wedge X_+) = E_*(X_+).$$

PROOF. First suppose that X has finite dimension, so that ν lifts to $\nu_n: X \rightarrow BF_n$ for suitably large n . Let $p_n: S(\nu_n) \rightarrow X$ be the associated n -sphere fibration, so that $T(\nu_n) = X \cup_{p_n} CS(\nu_n)$.

Now write $\Delta: T(\nu_n) \rightarrow T(\nu_n) \wedge X_+$ for the diagonal

$$x \mapsto \begin{cases} (x, p_n(x)), & x \neq \infty, \\ \infty, & x = \infty, \end{cases}$$

and consider the composite

$$\begin{aligned} \alpha(U): E \wedge T(\nu_n) &\xrightarrow{\text{id} \wedge \Delta} E \wedge T(\nu_n) \wedge X_+ \\ &\xrightarrow[\text{id} \wedge U \wedge \text{id}]{\rightarrow} E \wedge \Sigma^{n+1}E \wedge X_+ \xrightarrow[\mu \wedge \text{id}]{\rightarrow} E \wedge \Sigma^{n+1}X_+ \end{aligned}$$

where μ is the product in E .

Received by the editors August 4, 1980.

1980 *Mathematics Subject Classification.* Primary 55N20, 55P10.

¹ The first named author is supported in part by NSF Grant MCS 76-07051.

On homotopy groups this map induces a homomorphism $\phi_U: E_{*+n+1}(T(\nu_n)) \rightarrow E_*(X_+)$ which, by very construction, is $\cap U$. Thus ϕ_U is the usual Thom isomorphism, whence $\alpha(U)$ is an equivalence.

To complete the proof for all X , let $n \rightarrow \infty$.

COROLLARY 3. *Theorem 1 is true.*

PROOF. Let $X = \Omega$; choose $E = T(\nu)$ and U as the identity. Then $\alpha(\text{id}): T(\nu) \wedge T(\nu) \rightarrow T(\nu) \wedge \Omega_+$ is just the Thomification of $g': \Omega \times \Omega \rightarrow \Omega \times \Omega$ given by $g'(x, y) = (xy, y)$. Thus the inverse of $\alpha(\text{id})$ is the Thomification of g'^{-1} , and g'^{-1} is clearly g .

COROLLARY 4. *Suppose $\nu: X \rightarrow BO$ lifts to the $(i-1)$ -connected cover $BO\langle i \rangle$. Let $MO\langle i \rangle$ be the corresponding Thom spectrum. Then there is an equivalence $\beta: MO\langle i \rangle \wedge T(\nu) \rightarrow MO\langle i \rangle \wedge X_+$.*

Applying this, we deduce a result of [1].

COROLLARY 5. *Let $N(i)$ be such that $N(i)\xi$ is trivial over RP^i , where ξ is the Hopf line bundle. Then there is an equivalence $MO\langle i \rangle \wedge RP_{N(i)+j}^\infty \rightarrow MO\langle i \rangle \wedge \Sigma^{N(i)}RP_j^\infty$ for all j (where $RP_n^m = RP^m/RP^{n-1}$).*

PROOF. Consider the composite

$$\begin{aligned} MO\langle i \rangle \wedge RP_{N(i)+j}^\infty &\rightarrow MO\langle i \rangle \wedge RP_{N(i)}^\infty \wedge RP_j^\infty \\ &\xrightarrow[\cong]{\beta \wedge \text{id}} MO\langle i \rangle \wedge \Sigma^{N(i)}RP_+^\infty \wedge RP_j^\infty \rightarrow MO\langle i \rangle \wedge \Sigma^{N(i)}RP_j^\infty, \end{aligned}$$

where the first map is induced by the diagonal $T(N(i)\xi \oplus j\xi) \rightarrow T(N(i)\xi) \wedge T(j\xi)$, and the third by the projection $RP_+^\infty \rightarrow S^0$.

A simple homology calculation shows the composite to be an equivalence.

REFERENCES

1. D. M. Davis, S. Gitler, W. Iberkleid and M. Mahowald, *Orientability of bundles with respect to certain spectra*, Bol. Soc. Mat. Mexicana (to appear).
2. M. Mahowald, *Ring spectra which are Thom complexes*, Duke Math. J. **46** (1979), 549–559.

DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS 60201

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF MANCHESTER, MANCHESTER, ENGLAND M13 9PL