SHORTER NOTES

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A SHORT PROOF THAT A COMPACT ORDERED SPACE CANNOT BE MAPPED ONTO A NONMETRIC PRODUCT

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ABSTRACT. In this note we give an elementary proof of a theorem of Treybig: if a product $X \times Y$ of two infinite Hausdorff spaces is a continuous image of a compact linearly ordered space, then $X \times Y$ is metrizable.

It is the purpose of this paper to give a short proof of the well-known theorem [3] by L. B. Treybig, who shows that if the product of two compact infinite spaces is a continuous image of a compact ordered space,¹ then it is metrizable. In fact, we prove a stronger form of this theorem.

Let $\mathcal{R}$ be a family of closed subsets of a space X . We shall say that the elements of $\mathcal{R}$ are *separated by open neighbourhoods* if there exists a family $\mathcal{U}$ of open disjoint subsets of X such that each element of $\mathcal{R}$ is contained in some element of $\mathcal{U}$.

A map $f: X \rightarrow Y$ is said to be *almost infinitely covering* if there exists a countable family $\mathcal{F}$ of closed subsets of X separated by open neighbourhoods such that the set $\{P \in \mathcal{F}: y \in f(P)\}$ is infinite for all save a countable number of points $y \in Y$.

A sequence $\{P_n: n = 1, 2, \dots\}$ of subsets of an ordered space is said to be *increasing* (decreasing) if each element of P_n is less (greater) than each element of P_m for $n < m$.

Let A be a subset of an ordered space. A set $B \subset A$ is said to be a *convex component* of A if it is the maximal subset of A with respect to being convex.

LEMMA. *Let f be a continuous map from a compact ordered space K into a T_1 -space X . If $\mathcal{R}$ is a family of convex disjoint subsets of K , then the set $\{P \in \mathcal{R}: x, y \in f(P)\}$ is finite for every $x, y \in X, x \neq y$.*

PROOF. Let x and y be two distinct points of X . Suppose that there is an infinite subfamily $\mathcal{R}'$ of $\mathcal{R}$ such that $x, y \in f(P)$ for $P \in \mathcal{R}'$. The family $\mathcal{R}'$ consists of convex disjoint sets, so we can choose a sequence $\{P_n: n = 1, 2, \dots\} \subset \mathcal{R}'$, which

Received by the editors July 1, 1980.

1980 *Mathematics Subject Classification.* Primary 54F05, 54E35.

¹ By a *compact ordered space* we mean a linearly ordered set which is a compact space when equipped with the usual open-interval topology of the ordering.

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0002-9939/81/0000-0282/\$01.50

is either increasing or decreasing. Let p be the supremum (or, respectively, the infimum) of $\cup \{P_n: n = 1, 2, \dots\}$. Choose $a_n \in P_n \cap f^{-1}(\{x\})$ and $b_n \in P_n \cap f^{-1}(\{y\})$. Both the sequences $\{a_n: n = 1, 2, \dots\}$ and $\{b_n: n = 1, 2, \dots\}$ converge to p , but $f(a_n) = x$ and $f(b_n) = y$ for $n = 1, 2, \dots$, so f fails to be continuous at the point p . This yields a contradiction.

THEOREM. *If a Hausdorff space is a continuous image of a compact ordered space under an almost infinitely covering map, then it is metrizable.*

PROOF. Let f be a almost infinitely covering map from a compact ordered space K onto a Hausdorff space X . Let $\mathcal{F}$ be a countable family of closed subsets of X separated by open neighbourhoods and let D be a countable subset of X such that if $x \in X - D$ then the set $\{P \in \mathcal{F}: x \in f(P)\}$ is infinite. Put $L = \text{cl } \cup \mathcal{F}$. The set L is a closed subset of K , so it is a compact ordered space. Let $\mathcal{R}$ be the family of all convex components of the elements of $\mathcal{F}$ in L . Each $P \in \mathcal{F}$ is closed and open in L , so the family $\mathcal{R}$ is countable. The set $\{P \in \mathcal{R}: x, y \in f(P)\}$ is finite for every $x, y \in X$, $x \neq y$, in view of the Lemma. Hence, if $x \in X - D$ and $y \in X$, $y \neq x$, then there is a set $P \in \mathcal{R}$ such that $x \in f(P)$ and $y \notin f(P)$. Hence the family $\{\{x\}: x \in D\} \cup \{f(P_1) \cap \dots \cap f(P_n): P_1, \dots, P_n \in \mathcal{R}\}$ is a countable network for X , so, in view of a Theorem by Arhangel'skiĭ (see R. Engelking [1, Theorem 3.1.19]), the space X is metrizable.

COROLLARY (L. B. TREYBIG [3]). *Let X and Y be compact Hausdorff infinite spaces. If $X \times Y$ is a continuous image of a compact ordered space, then both X and Y are metrizable.*

PROOF. Let g be a map from a compact ordered space K onto $X \times Y$. The space Y is an infinite image of a compact ordered space, so it contains a converging sequence $\{a_n: n = 1, 2, \dots\}$. Denote $F_n = g^{-1}(X \times \{a_n\})$. Let p be a projection from $X \times Y$ onto X . Put $f = p \circ g$. The sets $F_1, F_2, \dots$ are separated by open neighbourhoods and $f(F_n) = X$ for $n = 1, 2, \dots$, so f is an almost infinite covering map. Hence X is metrizable in view of the Theorem. The proof for Y is analogous.

REMARK. There is no change either in the conclusion or in the proof of our Theorem if the assumption on f is replaced by a weaker one; that the set of all points $y \in Y$ for which the family $\{P \in \mathcal{F}: y \in f(P)\}$ is finite has a countable network. Another approach to Treybig's theorem is given in [2].

REFERENCES

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3. L. B. Treybig, *Concerning continuous images of compact ordered spaces*, Proc. Amer. Math. Soc. **15** (1964), 866–871.

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