A NOTE ON MATRICES WITH POSITIVE DEFINITE REAL PART

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ABSTRACT. A lemma characterizing complex matrices with positive definite real part is given. It is shown that many results concerning such matrices are straightforward consequences of this lemma.

Let A be an $n \times n$ complex matrix, let $H(A) = \frac{1}{2}(A + A^*)$ be its "real part" and let $S(A) = (A - A^*)/2i$ be its "imaginary part". Denote by Π_n the set of all $n \times n$ matrices A for which $H(A)$ is positive definite. A matrix D is strictly dissipative if $D = iA$ for $A \in \Pi_n$.

LEMMA. $A \in \Pi_n$ if and only if

$$A = T \operatorname{diag}(1 + i\alpha_1, \dots, 1 + i\alpha_n)T^*, \quad (*)$$

where $\alpha_j, j = 1, \dots, n$, are real numbers and T is a nonsingular matrix.

PROOF. If A is given by $(*)$, then $H(A) = TT^*$ is positive definite and so $A \in \Pi_n$.

If $A \in \Pi_n$, $(*)$ follows using the theory of pencils of quadratic forms [4, p. 313].

From $(*)$ follows: D is strictly dissipative if and only if

$$D = T \operatorname{diag}(i + \alpha_1, \dots, i + \alpha_n)T^*, \quad (*)'$$

where $\alpha_j, j = 1, \dots, n$, are real numbers and T is a nonsingular matrix.

The lemma was used long ago by Ostrowski and Taussky [7] and Taussky [8] and more recently by S. Friedland [3]. It was not used nor mentioned however in a series of papers [1], [2], [5], [6] dealing with matrices belonging to Π_n and with dissipative matrices, although most of the results obtained in these papers are straightforward consequences of the lemma. The purpose of this note is to revive this simple and somewhat forgotten lemma by applying it to derive some of these results, and by this showing its effectiveness in connection with matrices belonging to Π_n or dissipative matrices.

For $A \in \Pi_n$, the representation $(*)$ yields straightforward representations of various matrices connected with A . We bring now the representation of some of these matrices.

Let $A \in \Pi_n$ be represented by

$$A = T \operatorname{diag}(1 + i\alpha_1, \dots, 1 + i\alpha_n)T^*. \quad (*)$$

Then

(a) $A^* \in \Pi_n$ and

$$A^* = T \operatorname{diag}(1 - i\alpha_1, \dots, 1 - i\alpha_n)T^*.$$

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(b) A is nonsingular, $A^{-1} \in \Pi_n$ and

$$A^{-1} = (DT^{-1})^* \text{diag}(1 - i\alpha_1, \dots, 1 - i\alpha_n) DT^{-1},$$

where $D = \text{diag}(1/(1 + \alpha_1^2)^{1/2}, \dots, 1/(1 + \alpha_n^2)^{1/2})$.

(c) $A^{-1}A^*$ is similar to a unitary matrix and

$$A^{-1}A^* = (T^*)^{-1} \text{diag}\left(\frac{1 - i\alpha_1}{1 + i\alpha_1}, \dots, \frac{1 - i\alpha_n}{1 + i\alpha_n}\right) T^*.$$

(d) A^*A^{-1} is similar to a unitary matrix and

$$A^*A^{-1} = T \text{diag}\left(\frac{1 - i\alpha_1}{1 + i\alpha_1}, \dots, \frac{1 - i\alpha_n}{1 + i\alpha_n}\right) T^{-1}.$$

$$(e) \quad I + A^{-1}A^* = (T^*)^{-1} \text{diag}\left(\frac{2}{1 + i\alpha_1}, \dots, \frac{2}{1 + i\alpha_n}\right) T^*.$$

$$(f) \quad H(A) = TT^*.$$

$$(g) \quad S(A) = T \text{diag}(\alpha_1, \dots, \alpha_n) T^*.$$

$$(h) \quad [H(A)]^{-1}S(A) = (T^*)^{-1} \text{diag}(\alpha_1, \dots, \alpha_n) T^*.$$

Using the notation of [6], let $m = m(A) = \min_j \text{Re}(\lambda_j)$ and let $T = T(A) = \max_j (|\mu_j|)$, where λ_j , $j = 1, \dots, n$, are the eigenvalues of $A^{-1}A^*$ and μ_j , $j = 1, \dots, n$, are the eigenvalues of $[H(A)]^{-1}S(A)$. By (c) and (h),

$$m = \min_j \left((1 - \alpha_j^2) / (1 + \alpha_j^2) \right) \quad \text{and} \quad T = \max_j |\alpha_j|.$$

Hence,

$$m = (1 - T^2) / (1 + T^2),$$

and we got Theorem 3 of [6]. All the other results of [6] follow easily, using the representations of the various matrices given by (a)–(h). The results of [5], excluding §V, and the results of [1] and [2] concerning matrices in Π_n or dissipative matrices follow too. In particular, all the inertia theorems of [1] and [2] for matrices in Π_n or for dissipative matrices turn out to be results on the inertia of linear combinations of matrices having, by the representations (a)–(h), simultaneous congruent diagonalizations.

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