SEMRINGS SATISFYING PROPERTIES OF DISTRIBUTIVE TYPE

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ABSTRACT. Any distributive lattice admits a semiring structure in a natural way and this particular semiring satisfies some new distributive properties. The purpose of this note is to study whether a semiring with some or all of these new distributive properties admits the natural distributive lattice structure, and thus obtain Birkhoff's theorem as a corollary. Furthermore, characterizations of semirings satisfying some or all of these new distributive properties are obtained.

A triple $(S, +, \cdot)$ is a semiring if $(S, \cdot)$ and $(S, +)$ are semigroups, and $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for every a, b, c in S . Any distributive lattice $(S, \vee, \wedge)$ admits naturally a semiring structure $(S, +, \cdot)$ where $a + b = a \vee b$ and $ab = a \wedge b$ for every a, b in S . This semiring also satisfies the following conditions: for every a, b, c in S ,

- (A) $a + bc = (a + b)(a + c)$,
- (B) $bc + a = (b + a)(c + a)$,
- (C) $a = a + ab = a + ba$,
- (D) $a = ab + a = ba + a$.

The natural questions that arise are, whether some or all of the above conditions on a semiring imply that the semiring admits the above natural distributive lattice structure and whether these conditions are independent. Birkhoff [1, p. 135] provided necessary and sufficient conditions for a semiring containing a multiplicative identity to admit the above natural distributive lattice structure. Now, in this note we prove the same theorem for semirings not necessarily containing identity and characterize semirings with some or all of the above conditions in §2. We refer the reader to [2] for all the undefined terms in semigroup theory. '0' is called zero of a semiring $(S, +, \cdot)$ if $x \cdot 0 = 0 \cdot x = 0$ and $x + 0 = 0 + x = x$ for every x in S . e is called a multiplicative identity in $(S, +, \cdot)$ if $e \cdot x = x \cdot e = x$ for every x in S . A semigroup $(S, \cdot)$ is a left (right) zero semigroup if $a \cdot b = a$ ($a \cdot b = b$) for all a, b in S . $(S, \cdot)$ is a band if $a \cdot a = a$ for every a in S and is a rectangular band if $a \cdot b \cdot a = a$ for all a, b in S .

1. Relation between the conditions (A), (B), (C), (D). The following proposition provides a class of examples to show that condition (A) need not imply condition (B). These examples can be constructed very easily in the following way: consider any semigroup $(S, \cdot)$. Define right zero addition, i.e., $a + b = b$ for every a, b in S .

Received by the editors April 25, 1980 and, in revised form, August 12, 1980.

AMS (MOS) subject classifications (1970). Primary 16A78; Secondary 06A35.

Key words and phrases. Band, rectangular band, left zero semigroup, distributive lattice.

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0002-9939/81/0000-0305/\$02.50

Then $(S, +, \cdot)$ is a semiring. Direct verification yields

PROPOSITION 1.1. *Let $(S, +, \cdot)$ be a semiring in which $(S, +)$ is a right zero semigroup. Then the following are true:*

- (i) S satisfies condition (A).
- (ii) S satisfies condition (B) iff $(S, \cdot)$ is a band.
- (iii) S satisfies condition (D).
- (iv) If $|S| > 1$, S does not satisfy condition (C).
- (v) S does not contain 0 if $|S| > 1$.

EXAMPLE 1. Let $(S, \cdot)$ be a semigroup, which is not a band. Define left zero addition, i.e., $a + b = a$ for every a, b in S . Then $(S, +, \cdot)$ is a semiring and for every a, b, c in S , $a + bc = a$. If $a \neq a^2$, then $(a + b)(a + c) = a \cdot a \neq a$. Therefore S does not satisfy (A), but S satisfies (C).

In the following we shall prove the implications under some hypothesis.

PROPOSITION 1.2. *For a semiring $(S, +, \cdot)$ the following are true:*

- (i) If $(S, \cdot)$ is a band, then (C) implies (A).
- (ii) If S contains 0, (A) implies (C) and (B) implies (D).

PROOF. (i) For every a, b, c in S , we have

$$\begin{aligned} a + bc &= (a + ba) + bc = a + (ba + bc) = a + ac + (ba + bc) \\ &= a^2 + ac + ba + bc = (a + b)(a + c). \end{aligned}$$

Thus (A) is satisfied.

(ii) It suffices to prove that (A) implies (C). For every a in S , $a = a + 0 \cdot 0 = (a + 0)(a + 0) = a^2$. Now, if $a, b \in S$, we have

$$a = a + b \cdot 0 = (a + b)(a + 0) = (a + b)a = a^2 + ba = a + ba.$$

Similarly it can be proved $a = a + ab$.

The following example exhibits that (A) need not imply (B).

EXAMPLE 2. Let $S = \{0, a, e\}$. Set $e = e + e = e + a$ and $a = a + a = a + e = a \cdot a$. Let 0 and e act as zero and a multiplicative identity, respectively. Then the semiring $(S, +, \cdot)$ satisfies (A) but (B) does not hold because $e = 0 + e = a \cdot 0 + e$, whereas $(a + e)(0 + e) = ae = a$.

In the presence of zero, we will show later (in 2.8) that (A) implies (B) iff $(S, +)$ is a commutative semigroup.

Now we study how the additive and multiplicative structures are related under these new conditions on a semiring. In any semiring $(S, +, \cdot)$, if $(S, \cdot)$ is a rectangular band, then $(S, +)$ is a band, since for every x in S , we have $x = x(x + y)x = x^3 + xyx = x + x$. If, in addition, the above semiring satisfies condition (C), we have the following:

PROPOSITION 1.3. *Let $(S, +, \cdot)$ be a semiring in which $(S, \cdot)$ is a rectangular band. Then S satisfies condition (C) iff $(S, +)$ is a left zero semigroup.*

PROOF. Let S satisfy (C). From the above, we note that $(S, +)$ is a band. Then for every a and b in S , we have

$$\begin{aligned} a &= a + ab = a + (a + ba)b = a + (ab + bab) \\ &= a + (ab + b) = (a + ab) + b = a + b. \end{aligned}$$

The converse is a dual statement of 1.1(iii).

PROPOSITION 1.4. *Let $(S, +, \cdot)$ be a semiring satisfying condition (C). Then the following are true:*

- (i) *If $(S, \cdot)$ is a band, then $(S, +)$ is a band.*
- (ii) *If S satisfies (A), then $(S, +)$ and $(S, \cdot)$ are both bands.*

PROOF. (i) Setting $b = a$ in condition (C), we have $a = a + a^2 = a + a$.

(ii) Clearly $a = a + a \cdot a = a + a^2$. Then by (A),

$$a = a + a \cdot a = (a + a)(a + a).$$

Therefore $a = 4a^2$. Then $5a = 4a + a = 4a + 4a^2 = 4a$, which implies $8a = 5a + 3a = 4a + 3a = 7a$. Continuing in this way, we have $8a = 4a$. Thus for every a in S , $4a$ is an idempotent. Therefore $a = 4a^2$ is an additive idempotent. Thus $(S, +)$ is a band. Now

$$a = a + a^2 = (a + a)(a + a) = a \cdot a = a^2.$$

Hence $(S, \cdot)$ is also a band.

By Proposition 1.2, (C) implies (A) if $(S, \cdot)$ is a band. By Proposition 1.4, if $(S, +, \cdot)$ satisfies both (A) and (C), then $(S, +)$ and $(S, \cdot)$ are both bands. So it may appear that if $(S, +)$ and $(S, \cdot)$ are both bands in a semiring $(S, +, \cdot)$ satisfying condition (A), then $(S, +, \cdot)$ also satisfies condition (C). This can be seen to be false in the example of a semiring $(S, +, \cdot)$ where $(S, \cdot)$ is a band and $(S, +)$ is a right zero semigroup by virtue of Proposition 1.1(iv).

2. Structure theorems.

THEOREM 2.1. *Let $(S, +, \cdot)$ be a semiring satisfying any one of the following conditions:*

- (i) *$(S, \cdot)$ is a band and S satisfies condition (C).*
- (ii) *S satisfies conditions (A) and (C).*
- (iii) *S contains 0 and S satisfies (A).*

Then S is a semilattice of semirings S_α , where $(S_\alpha, \cdot)$ is a rectangular band and $(S_\alpha, +)$ is a left zero semigroup if the multiplicative completely prime ideals are closed under addition.

PROOF. By virtue of Propositions 1.4(iii) and 1.2(ii), (ii) and (iii) separately imply (i). So we may assume (i). Now by 4.7 of [2, p. 44], the band $(S, \cdot)$ admits a decomposition $S = \bigcup_{\alpha \in I} S_\alpha$, where I is a semilattice and each $(S_\alpha, \cdot)$ is a rectangular band. In fact $(S_\alpha, +, \cdot)$ is a semiring. For this, it suffices to show that whenever $x, y \in S_\alpha$, $x + y \in S_\alpha$. As noted in [2, p. 29] two elements a and b of S belong to the same (N -class) S_α iff for every completely prime ideal P either both a and b belong to P or neither of them belongs to P . If P is a completely prime ideal

in $(S, \cdot)$ containing x and y , then $x + y \in P$ since P is closed under addition by hypothesis. Conversely let P be a completely prime ideal in $(S, \cdot)$ such that $x + y \in P$. Since $x = x + xy = x(x + y)$, $x \in P$. Hence $x + y \in S_\alpha$. Thus $(S_\alpha, +, \cdot)$ is a semiring. Since each S_α also satisfies (C) and $(S_\alpha, \cdot)$ is a rectangular band, $(S_\alpha, +)$ is a left zero semigroup by Proposition 1.3.

NOTE. The converse of the above theorem need not be true even if the semilattice is replaced by a chain. Let $S = \{x, y\}$. Define $x = x + x = x \cdot x$ and $y = y + y = y \cdot y = x + y = y + x = xy = yx$. $(S, +, \cdot)$ is a semiring. Also $S = \{x\} \cup \{y\}$, where each of these sets is a semiring with the properties described in the theorem. But S does not satisfy condition (C) since $x + xy = x + y = y \neq x$. Also S does not contain 0.

PROPOSITION 2.2. *Let $(S, +, \cdot)$ be a semiring satisfying condition (C). If $(S, +)$ and $(S, \cdot)$ are semilattices, then S admits a lattice structure by defining: for every a, b in S , $a \leq b$ iff $a + b = b$ and $a = ab$; $a \vee b = a + b$ and $a \wedge b = ab$. Moreover this is the only ordering satisfying $a \vee b = a + b$ and $a \wedge b = ab$ for every a, b in S and the lattice structure is distributive.*

PROOF. Clearly ' $\leq$ ' is a partial order. If $a \leq c$ and $b \leq c$, then $a + c = c = b + c$, $a = ac$, and $b = bc$. Then

$$c = c + c = (a + c) + (b + c) = (a + b) + c + c = a + b + c$$

and $a + b = ac + bc = (a + b)c$. Hence $a + b \leq c$ by the definition of the order and thus $a \vee b = a + b$. Similar verification yields $a \wedge b = ab$. by the distributive property of multiplication over addition we have $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $(b \vee c) \wedge a = (b \wedge a) \vee (c \wedge a)$ for every a, b, c in S . Now

$$\begin{aligned}(a + b)(a + c) &= a^2 + ac + ba + bc = (a + ac) + ba + bc \\ &= (a + ba) + bc = a + bc\end{aligned}$$

by condition (C). Thus for every a, b, c in S we have $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and, similarly, $(b \wedge c) \vee a = (b \vee a) \wedge (c \vee a)$. The remainder of the proof is now evident.

THEOREM 2.3. *Let $(S, +, \cdot)$ be a semiring satisfying conditions (C) and (D). If $(S, \cdot)$ is a band, then $(S, +)$ and $(S, \cdot)$ are semilattices.*

PROOF. By 1.4(i), $(S, +)$ is a band. Now it suffices to prove that $(S, +)$ and $(S, \cdot)$ are both commutative. For every a, b in S , we have $a(a + b) = a^2 + ab = a + ab = a$ and $a(b + a) = ab + a^2 = ab + a = a$. Therefore $b = b(b + a)$. This implies

$$\begin{aligned}a + b &= [a(b + a)] + [b(b + a)] = (a + b)(b + a) \\ &= (a + b)b + (a + b)a = ab + b + a + ba = b + a.\end{aligned}$$

Hence $(S, +)$ is commutative. Also,

$$\begin{aligned}ab &= (a + ba)(b + ba) = a(b + ba) + ba(b + ba) \\ &= ab + aba + bab + baba = ab + ba.\end{aligned}$$

Interchanging a and b , we have $ba = ab + ba$. Since $(S, +)$ is commutative, this implies $ab = ba$. Thus $(S, \cdot)$ is commutative.

THEOREM 2.4. *Let $(S, +, \cdot)$ be a semiring. Then S admits a lattice structure with $a \vee b = a + b$ and $a \wedge b = ab$ for every a, b in S iff S satisfies either one of the following equivalent conditions:*

- (i) S satisfies conditions (A), (C) and (D).
- (ii) $(S, \cdot)$ is a band and S satisfies conditions (C) and (D).

In this case the lattice is distributive.

PROOF. The above two conditions are equivalent by virtue of 1.2(i) and 1.4(ii). Let S admit a lattice structure with $a \vee b = a + b$ and $a \wedge b = ab$ for every a, b in S . Then clearly $(S, +)$ and $(S, \cdot)$ are both semilattices. Moreover by 2.2, $a \leq b$ if $a + b = b$ and $ab = a$. Since $ab = a \wedge b \leq a$, we have $ab + a = a$. By commutativity, it follows that S satisfies conditions (C) and (D). The converse is evident by combining 1.4(ii), 2.2 and 2.3.

COROLLARY 2.5 [BIRKHOFF]. *Let $(S, +, \cdot)$ be a semiring containing the multiplicative identity e . Then S admits a distributive lattice structure with $a + b = a \vee b$ and $ab = a \wedge b$ for every a, b in S if $(S, \cdot)$ is a band and $e + a = a + e = e$ for every a in S .*

PROOF. This is evident from 2.4 by noting for every a, b in S , $a = ea = (e + b)a = a + ba$ and $a = a(e + b) = a + ab$ and, hence, condition (C) is satisfied. Similarly it can be proved that condition (D) is satisfied.

By virtue of 1.2(ii) and its dual statement, we have

COROLLARY 2.6. *Let $(S, +, \cdot)$ be a semiring containing 0. If S satisfies conditions (A) and (B), then S admits a distributive lattice structure with $a \vee b = a + b$ and $a \wedge b = ab$ for every a, b in S .*

COROLLARY 2.7. *Let $(S, +, \cdot)$ be a semiring containing 0. If S satisfies condition (A) and if $(S, +)$ is commutative, then S admits a distributive lattice structure defined as in 2.2.*

PROOF. Since $(S, +)$ is commutative, condition (A) implies condition (B). Then the conclusion is obvious by 2.6.

COROLLARY 2.8. *Let $(S, +, \cdot)$ be a semiring containing 0 and satisfying condition (A). Then S satisfies condition (B) iff $(S, +)$ is commutative.*

PROOF. If $(S, +)$ is commutative, evidently (A) implies (B). If S satisfies condition (B), S admits a distributive lattice structure with $a \vee b = a + b$ for every a, b in S by 2.6. Since $a \vee b = b \vee a$, $a + b = b + a$ for every a, b in S . Thus $(S, +)$ is commutative.

The above corollary need not be true if the commutativity of $(S, +)$ is replaced by the commutativity of $(S, \cdot)$. Example 2 proves this assertion.

Finally we conclude that Theorem 2.4 is false if we omit either condition (A) in (i) or the condition that $(S, \cdot)$ is a band in (ii). Let $(S, +)$ be an additive group

adjoined with 0. Define $x \cdot y = 0$ for every x, y in S . Then $(S, +, \cdot)$ is a semiring satisfying conditions (C) and (D). $(S, \cdot)$ is not a band and also S does not satisfy condition (A). Clearly $(S, +, \cdot)$ does not admit the above natural distributive lattice structure since $(S, \cdot)$ fails to be a band.

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