A REMARK ON THE SPACES $V_{\Lambda, \alpha}^p$

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ABSTRACT. A function $f \in L^p$, $p > 1$, over an interval in R^n , is in $V_{\Lambda, \alpha}^p$ if, corresponding to the i th coordinate direction, $i = 1, \dots, n$, there is an equivalent function which is of Λ -bounded variation on a.e. line l_i in that direction and whose Λ -variation on those lines is in L^α , $\alpha > 1$, as a function of the other $(n - 1)$ variables. For each i , another equivalent function may be chosen so that on a.e. l_i it has an internal saltus at each point. It is shown that for this function, the Λ -variation on the lines l_i is a measurable function of the other variables. This was known for $n = 2$; for $n > 2$, the measurability was previously assumed as an additional hypothesis. The classes $V_{\Lambda, \alpha}^p$ are Banach spaces and have been shown to be of interest in the study of localization of multiple Fourier series.

In [2], [3], a new class of Banach spaces was introduced in connection with the localization problem for double Fourier series. The definition of these spaces involves a notion of Λ -variation for functions of several variables which is an extension of bounded variation in the sense of Cesari. If the Λ -variation along lines in one coordinate direction is considered to be a function of the remaining variables, its measurability was shown in the 2 dimensional case by an argument which does not seem to apply to higher dimensions. For the usual variation, the measurability follows easily from the fact that refining a partition increases the corresponding estimate of the variation. This property is not shared by the Λ -variation.

Let $\Lambda = \{\lambda_n\}$ be a nondecreasing sequence of positive numbers such that $\sum 1/\lambda_n$ diverges. A real function f on an interval $[a, b]$ is of Λ -bounded variation [Λ BV] if $\sum_1^\infty |f(a_n) - f(b_n)|/\lambda_n$ converges for every sequence $\{[a_n, b_n]\}$ of nonoverlapping intervals in $[a, b]$. The supremum of such sums is necessarily finite and is called the Λ -variation of f .

In [2], [3], the class $V_{\Lambda, \alpha}^p$ was defined to consist of those $f \in L^p$, $p > 1$, on an interval in R^n , to which there correspond equivalent f_i , $i = 1, \dots, n$, such that on a.e. line l_i in the i th coordinate direction, $f_i \in \Lambda$ BV and the Λ -variation of f_i on l_i is in L^α , $\alpha > 1$, as a function of the remaining $(n - 1)$ variables. For $n > 2$, it was assumed further that each f_i , restricted to almost any l_i , has an internal saltus at each point, i.e., at each point its value is between its upper and lower limits.

The result of this paper implies that the additional requirement made in this definition in the case $n > 2$ is superfluous.

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It was shown in [3] that if $f_i \in \Lambda BV$ on a.e. l_i , then there is an equivalent function $\tilde{f}_i$ which is in ΛBV on a.e. l_i and which has an internal saltus at each point of a.e. l_i . In [5], it was shown that if a function in ΛBV is altered at points of discontinuity so that it has an internal saltus at each such point, then its Λ -variation is minimized. For $n = 2$, it was shown in [3] that $V(\tilde{f}_i)$, the Λ -variation of $\tilde{f}_i$ on l_i , is a measurable function. We shall show here that this is valid for $n > 2$ as well, obviating the need for the additional requirement previously made in this case. Then for $n \geq 2$, $V(\tilde{f}_i) \in L^\alpha$ and $V_{\Lambda, \alpha}^p$ is a Banach space with norm

$$\|f\|_p + \sum_1^n \|V(\tilde{f}_i)\|_\alpha.$$

In [3] it was shown that the class $V_{H,1}^1$, where $H = \{1/n\}$, plays a central role in the localization of double Fourier series by square and rectangular sums.

Suppose $J = \prod_1^n [a_i, b_i]$ is an interval in R^n and let

$$l_i = l_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$$

be the (nonempty) intersection of J and the line through $(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)$ in the direction of the i th coordinate axis. If $f: J \rightarrow R^1$, $V_i(f)$ will denote the Λ -variation of f on an l_i , so that $V_i(f)$ is a function of the $(n-1)$ variables x_j , $j \neq i$. We may set $V_i(f) = \infty$ for those l_i on which $f \notin \Lambda BV$. When f restricted to l_i is continuous at x , it is said that f is *linearly continuous* at x on l_i . Analogous terminology is employed if f is right or left continuous at x or discontinuous at x .

A function f is in class V_Λ if it is measurable and there exist *corresponding functions* f_i , $i = 1, \dots, n$, equivalent to f , such that $V_i(f_i) < \infty$ a.e. for every i .

We may now state our principal result.

THEOREM. *Let $f \in V_\Lambda$ and f_i a corresponding function which has an internal saltus at each of its points of linear discontinuity on the l_i for which $V_i(f_i) < \infty$. Then $V_i(f_i)$ is a measurable function.*

We note that the requirement that f_i have an internal saltus on an l_i implies that f_i is linearly right continuous at the left endpoint of l_i and linearly left continuous at the right endpoint of l_i .

A real function defined on a set $E \subset R^n$ is *approximately lower semicontinuous* at a point $x_0 \in E$ if, for any $\epsilon > 0$, there is a set $A \subset E$ having density 1 at x_0 and such that $f(x) > f(x_0) - \epsilon$ for $x \in A$.

We shall prove the following lemma.

LEMMA. *If f is approximately lower semicontinuous a.e. in E , then f is measurable on E .*

A stronger form of this lemma holds in which the hypothesis that A have density 1 at x_0 is replaced by the requirement that the lower inner density of A at x_0 be positive. This result of Kamke [4] is not well-known and so we include a proof of the lemma based on the method of Saks employed by Kamke.

PROOF OF LEMMA. Choose a real number c . Let $H = \{x | f(x) > c\} \cap E$. We must show that H is measurable. Choose $x_0 \in H$ such that f is approximately lower

semicontinuous at x_0 and let $\varepsilon = f(x_0) - c$. Then there is a set A having density 1 at x_0 such that $x \in A$ implies $f(x) > f(x_0) - \varepsilon = c$. If I is an interval containing x_0 and m_i denotes inner Lebesgue measure, then

$$m_i(H \cap J) > m_i(A \cap J)$$

implying that the density of H at x_0 is 1. Write $H = H_1 \cup H_2$ such that $H_1 \cap H_2 = \emptyset$, H_1 is measurable, and $m_i(H_2) = 0$. The density of H_1 equals 1 at a.e. point of H_1 and equals 0 at a.e. point of H_2 . Since $m_i(H_2) = 0$, the lower inner density of H equals 0 at a.e. point of H_2 . Since the density of H equals 1 at a.e. point of H , H_2 must be a set of measure zero. $\square$

We may now establish our principal result.

PROOF OF THEOREM. From our lemma we see that our theorem will be established if we can show that $V_i(f_i)$ is approximately lower semicontinuous a.e. in $\prod_{j \neq i} [a_j, b_j]$.

Without loss of generality, we assume $\lambda_1 \geq 1$, $i = 1$, and write $x_1 = x$, $a_1 = a$, $b_1 = b$, $(x_2, \dots, x_n) = y$, $f_1 = f$, and denote the domain of f by $[a, b] \times I$. $V(y)$ will denote the value of $V_1(f_1)$ on the I_1 determined by y .

We note that at a.e. y in I , f is approximately continuous in y for a.e. x in $[a, b]$ [1, 3.1.3(4)]. Choose such a y_0 for which $V(y_0) < \infty$.

Let $\varepsilon > 0$ be given. There are nonoverlapping intervals $[a_j, b_j]$, $j = 1, \dots, k$, in $[a, b]$ such that

$$\sum_{j=1}^k |f(a_j, y_0) - f(b_j, y_0)| / \lambda_k > V(y_0) - \varepsilon/2.$$

Note that in estimating the Λ -variation, we may choose the a_j and b_j to be points of linear continuity of f , since $f(x, y_0)$ has an internal saltus at each point of linear discontinuity [5]. Since f is approximately continuous in y a.e. in $[a, b]$, the a_j and b_j can be chosen so that $f(a_j, y)$ and $f(b_j, y)$ are approximately continuous in y at y_0 for $j = 1, \dots, k$ and the above inequality is valid.

For each j , there are sets A_j and B_j in I , each having density 1 at y_0 such that $|f(a_j, y_0) - f(a_j, y)| < \varepsilon/4k$ for $y \in A_j$ and $|f(b_j, y_0) - f(b_j, y)| < \varepsilon/4k$ for $y \in B_j$. Then $C = \bigcap_{j=1}^k (A_j \cap B_j)$ has density 1 at y_0 and

$$V(y) \geq \sum_{j=1}^k |f(a_j, y) - f(b_j, y)| / \lambda_k > V(y_0) - \varepsilon \quad \text{for } y \in C.$$

Thus V is approximately lower semicontinuous a.e. in I and is, therefore, measurable. $\square$

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