ON THE EXISTENCE OF GLOBAL SOLUTIONS OF THE NONLINEAR PARABOLIC EQUATION OF EELLS-SAMPSON OVER PRODUCT MANIFOLDS

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ABSTRACT. We extend results of Lemaire and Hamilton on the existence of global solutions of the equation in the title to warped product manifolds with boundaries.

1. Introduction. The object of this note is to prove the following

THEOREM 1. *Let $M = N \times_{\psi} K$ and $M' = N' \times_{\psi} K'$ be warped products of compact Riemannian manifolds; N, N' being with boundary and K, K' without boundary. Suppose N' has nonpositive sectional curvature everywhere and convex boundary $\partial N'$. Let $F: K \rightarrow K'$ be a harmonic map with constant energy density and $f_0: N \rightarrow N'$ be a C^{∞} map. Let $u: \partial M \times [0, \infty) \rightarrow M'$ be a C^{∞} map independent of time $t \in [0, \infty)$ and suppose $u = f_0 \times F$ on the corner $\partial M \times 0$. Then the Dirichlet initial-boundary value problem for the nonlinear parabolic equation of Eells-Sampson*

$$\begin{aligned}\partial U / \partial t &= \tau(U) \quad \text{on } M \times [0, \infty), \\ U &= f_0 \times F \quad \text{on } M \times 0, \\ U &= u \quad \text{on } \partial M \times [0, \infty)\end{aligned}\tag{D}$$

has a global solution $U: M \times [0, \infty) \rightarrow M'$, which is unique, and C^{∞} except on the corner $\partial M \times 0$.

For the terminology and the notation in the theorem, see §2.

In [3], Hamilton proved that if M' has nonpositive sectional curvature everywhere, then the Dirichlet problem (D) has a global solution $U(x, t) = U_t(x): M \times [0, \infty) \rightarrow M'$ for arbitrary initial condition $U_0: M \times 0 \rightarrow M'$ a C^{∞} map with $U_0 = u$ on $\partial M \times 0$, and $\{U_t\}_t$ converges uniformly to a harmonic map $U_{\infty}: M \rightarrow M'$, $\tau(U_{\infty}) = 0$ as $t \rightarrow \infty$.

In contrast with this, Theorem 1 shows that for a special class of initial conditions the Dirichlet problem (D) has a global solution even when the target manifold M' admits positive sectional curvature, for the warped product $M' = N' \times_{\psi} K'$ has positive sectional curvature by a suitable choice of scale factor ψ . It is, however, not known in general whether the solution $\{U_t\}_t$ subconverges to a harmonic map.

Received by the editors April 16, 1980.

1980 *Mathematics Subject Classification.* Primary 35K55; Secondary 58E20.

Key words and phrases. Nonlinear parabolic equation of Eells-Sampson, harmonic map, warped product.

¹Partially supported by the Sakkokai Foundation.

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0002-9939/81/0000-0311/\$02.25

In the case where N and N' are without boundary, Theorem 1 was proved by Lemaire [4]. I wish to thank him for suggesting this investigation and helpful discussion.

2. Preliminaries. Let $M = (M, g)$ and $M' = (M', g')$ be compact Riemannian manifolds and $U: M \rightarrow M'$ a C^∞ map. Then the *energy density* $e(U)$ of U is defined by $e(U) = \frac{1}{2}|dU|^2$, $|dU|$ being the norm of the differential dU of U , and the *energy* $E(U)$ of U is defined by the integral

$$E(U) = \int_M e(U) d\mu_M.$$

U is called *harmonic* if it satisfies the Euler-Lagrange equation $\tau(U) = 0$, which is a semilinear elliptic system of second order, of the energy functional E . The Euler-Lagrange operator $\tau(U) = \text{Div}(dU)$ defines a C^∞ vector field, called the *tension field* of U , on M' along U and is expressed in local coordinates (x^i) , (y^a) of M and M' as

$$\tau(U)^\alpha = \Delta U^\alpha + g^{ij} {}_M \Gamma_{\beta\gamma}^\alpha(U) U_i^\beta U_j^\gamma,$$

where Δ is the Laplacian on M , ${}_M \Gamma_{\beta\gamma}^\alpha$ are Christoffel symbols on M' and $U_i^\alpha = \partial U^\alpha / \partial x^i$. See Eells-Sampson [2] for details.

Let $N = (N, g)$ and $K = (K, h)$ be Riemannian manifolds and φ a positive C^∞ function on N . Then their *warped product* $N \times_\varphi K$ is defined to be the Riemannian manifold $(N \times K, g \times \varphi h)$, $g \times \varphi h$ being the product metric of g and φh . For the fundamental properties of warped product manifolds, see Bishop-O'Neill [1].

For a Riemannian manifold N with boundary ∂N , we say that ∂N is *convex* if the second fundamental form of ∂N in N with respect to the inward unit normal vector is positive semidefinite everywhere.

For C^∞ maps $f: N \rightarrow N'$ and $F: K \rightarrow K'$, $f \times F$ denotes their product, the map $f \times F: N \times K \rightarrow N' \times K'$ defined by $f \times F(x_1, x_2) = (f(x_1), F(x_2))$ for $x_1 \in N$, $x_2 \in K$.

We use the Sobolev spaces $L_k^p(M \times [\alpha, \beta])$ with inhomogeneous weights in variables on a compact product Riemannian manifold $M \times [\alpha, \beta]$. For each even integer $k > 0$ and each real number $1 < p < \infty$, $L_k^p(M \times [\alpha, \beta])$ is the Sobolev space of tempered distributions on $M \times [\alpha, \beta]$ whose first k -derivatives in the space direction M and first $(k/2)$ -derivatives in the time direction $[\alpha, \beta]$ are L^p -integrable. For other real numbers $k \in \mathbb{R}$ we refer to Hamilton [3] for the precise definition. $L_k^p(M \times [\alpha, \beta], \mathbb{R}^N)$ denotes the N -fold direct sum of $L_k^p(M \times [\alpha, \beta])$ with itself.

3. Proof of Theorem 1. To prove Theorem 1 we follow the method of Hamilton [3] and modify the argument in Lemaire [4].

We first imbed M' into a suitable Euclidean N -space $\mathbb{R}^N$ with metric defined as in [3, §IV, 5] so that the composition of U with the imbedding also satisfies (D). It is known in [3, §IV, 11] that for a given initial condition the Dirichlet problem (D) has a local solution $U: M \times [0, \varepsilon] \rightarrow M'$, $\varepsilon > 0$, of class L_2^p ($p > \dim M' + 2$), which is unique, and C^∞ except on the corner $\partial M \times 0$.

Let $U: M \times [0, \omega) \rightarrow M'$ be a maximal solution of the Dirichlet problem (D). Our goal is to prove $\omega = \infty$. To this end, we suppose $\omega < \infty$ and will prove that all the derivatives of U remain uniformly bounded on $M \times [0, \omega)$. Then U extends smoothly to a solution on $M \times [0, \omega]$ and hence to that on $M \times [0, \omega + \varepsilon]$, $\varepsilon > 0$. This shows that U is not maximal and ω must be infinity in consequence.

In what follows, we denote various constants independent of $t \in [0, \omega)$ by the same letter C and covariant differentiation by ∇ . Set $e = e(U)$ and $E = E(U)$. Define $\kappa = \frac{1}{2}|\partial U/\partial t|^2$ and $H = \int_M \kappa d\mu_M$. Then, since the energy density $e(F)$ of F is assumed to be constant and the sectional curvature of N' nonpositive, we have

LEMMA 1 [4]. (i) For all $t \in [0, \omega)$ the solution U is of the form $U = f_t \times F$ with $f_t: N \times t \rightarrow N'$.

(ii) $\partial e/\partial t \leq \Delta e + C_1 e + C_1 - |\nabla \nabla U|^2$ on $M \times [0, \omega)$.

(iii) $\partial \kappa/\partial t \leq \Delta \kappa + C_2 \kappa - |\nabla \partial U/\partial t|^2$ on $M \times [0, \omega)$.

Let ν be the outward unit normal vector of ∂M in M and $\partial/\partial \nu$ denote the normal derivative. Then from the Dirichlet boundary condition we have

LEMMA 2 [3, §V, 3, 4]. (i) $dE/dt = -2H$ for all $t \in [0, \omega)$.

(ii) $\partial e/\partial \nu = 2me + a\nabla \nabla U + b\nabla U$ on $\partial M \times [0, \omega)$, where $a\nabla \nabla U + b\nabla U$ is a second order linear operator with C^∞ coefficients depending only on $U|_{\partial M \times [0, \omega)}$ $= u$ and m is the mean curvature of ∂M in M .

(iii) $\kappa = 0$ and $\partial \kappa/\partial \nu = 0$ on $\partial M \times [0, \omega)$.

Set $\Theta = \kappa \exp(-C_2 t)$ and $T = \int_M \Theta d\mu_M = H \exp(-C_2 t)$. Then $\partial \Theta/\partial t \leq \Delta \Theta$ by Lemma 1(iii) and $\partial \Theta/\partial \nu = 0$ by Lemma 2(iii). Using Green's theorem

$$\frac{\partial T}{\partial t} = \int_M \frac{\partial \Theta}{\partial t} d\mu_M < \int_M \Delta \Theta d\mu_M = \int_M \frac{\partial \Theta}{\partial \nu} d\mu_{\partial M} = 0.$$

Thus T is monotone decreasing and hence $T \leq T_0$, that is $H \leq H_0 \exp(C_2 t)$. Since $dE/dt = -2H \leq 0$ by Lemma 2(i), E is monotone decreasing, $E \leq E_0$ and $-dE/dt \leq 2H_0 \exp(C_2 t)$. From Lemma 1(ii) we have

$$\frac{dE}{dt} = \int_M \frac{\partial e}{\partial t} d\mu_M < \int_M \Delta e d\mu_M - \int_M |\nabla \nabla U|^2 d\mu_M + C,$$

and from Lemma 2(ii)

$$\int_M \Delta e d\mu_M = \int_{\partial M} \frac{\partial e}{\partial \nu} d\mu_{\partial M} < C \left\{ \int_{\partial M} |\nabla U|^2 d\mu_{\partial M} + \int_{\partial M} |\nabla \nabla U| d\mu_{\partial M} + 1 \right\}.$$

It follows from these that

$$\int_M |\nabla \nabla U|^2 d\mu_M < C \left\{ \int_{\partial M} |\nabla U|^2 d\mu_{\partial M} + \int_{\partial M} |\nabla \nabla U| d\mu_{\partial M} + \exp(Ct) \right\}. \quad (1)$$

Now fix a $\delta > 0$ with $0 < \delta < \omega/4$. We will estimate the Sobolev norms of U on $M \times [t, t + \delta]$ ($\delta \leq t < \omega - \delta$). We write simply $\|U\|_{L^p(M \times [t, t + \delta])}$ for $\|U\|_{L^p(M \times [t, t + \delta], \mathbb{R}^N)}$.

L_2^2 norm. Recall that Image U , the image of U in $\mathbf{R}^N$, is bounded, $E < E_0$ and $H < H_0 \exp(C_2 t)$. Then we get from (1) that

$$\begin{aligned} \|U\|_{L_2^2(M \times [t, t+\delta])}^2 &= \int_{M \times [t, t+\delta]} \left(|U|^2 + |\nabla U|^2 + |\nabla \nabla U|^2 + \left| \frac{\partial U}{\partial t} \right|^2 \right) d\mu_M dt \\ &\leq C \int_{\partial M \times [t, t+\delta]} (|\nabla U|^2 + |\nabla \nabla U|) d\mu_{\partial M} dt + C \exp(Ct). \end{aligned} \quad (2)$$

It is known in [3, §V, 6, 8] that

$$\begin{aligned} \int_{\partial M \times [t, t+\delta]} |\nabla U|^2 d\mu_{\partial M} dt &\leq C \|U\|_{L_2^2(M \times [t, t+\delta])}^{4/3}, \\ \int_{\partial M \times [t, t+\delta]} |\nabla \nabla U| d\mu_{\partial M} dt &\leq C(1 + \|U\|_{L_2^2(M \times [t, t+\delta])})^{3/2}. \end{aligned} \quad (3)$$

From (2) and (3) it then follows, as in the proof of the theorem in [3, §IV, 8], that

$$\|U\|_{L_2^2(M \times [t, t+\delta])} \leq C \exp(Ct). \quad (4)$$

L_2^2 norm. Set $\Lambda = e + 1$. Then, by Lemmas 1(ii) and 2(ii), Λ satisfies the inequalities

$$\begin{aligned} \partial \Lambda / \partial t &\leq \Delta \Lambda + C_1 \Lambda && \text{on } M \times [0, \omega), \\ \partial \Lambda / \partial \nu &\leq 2m\Lambda + a \nabla \nabla U + b \nabla U - 2m && \text{on } \partial M \times [0, \omega). \end{aligned}$$

We define an auxiliary function λ_θ as the solution of the following linear parabolic initial-boundary value problem:

$$\begin{aligned} \frac{\partial \lambda_\theta}{\partial t} &= \Delta \lambda_\theta + C_1 \lambda_\theta && \text{on } M \times [\theta - \delta, \theta + \delta], \\ \lambda_\theta &= \Lambda && \text{on } M \times \{\theta - \delta\}, \\ \frac{\partial \lambda_\theta}{\partial \nu} &= 2m\lambda_\theta + a \nabla \nabla U + b \nabla U - 2m && \text{on } \partial M \times [\theta - \delta, \theta + \delta]. \end{aligned} \quad (5)$$

The problem (5) has a unique classical solution on $M \times [\theta - \delta, \theta + \delta]$ [3, §IV, 8]. It follows from the maximum principle [3, §V, 9] applied to $\Lambda - \lambda_\theta$ that $\Lambda \leq \lambda_\theta$ on $M \times [\theta - \delta, \theta + \delta]$.

By the L^1 form of the maximum principle [3, §V, 9] applied to (5) we have

$$\begin{aligned} \|\lambda_\theta\|_{L^1(M \times [\theta - \delta, \theta + \delta])} &\leq C(\|\Lambda\|_{L^1(M \times \{\theta - \delta\})} + \|a \nabla \nabla U + b \nabla U - 2m\|_{L^1(\partial M \times [\theta - \delta, \theta + \delta])}), \end{aligned}$$

from which we get

$$\|\lambda_\theta\|_{L^1(M \times [\theta - \delta, \theta + \delta])} \leq C \exp(C\theta)$$

due to (3) and (4). On the other hand, by the dual version of the Sobolev imbedding theorem [3, §V, 9] we have, for $k > p + 2$, $\|\lambda_\theta\|_{L_k^2(M \times [\theta - \delta, \theta + \delta])} \leq C \|\lambda_\theta\|_{L^1(M \times [\theta - \delta, \theta + \delta])}$. Hence for $k > p + 2$, $\|\lambda_\theta\|_{L_k^2(M \times [\theta - \delta, \theta + \delta])} \leq C \exp(C\theta)$. Then, by a mimicry of the proof of the theorem in [3, §V, 11] it is verified that

$$\|U\|_{L_2^2(M \times [t, t+\delta])} \leq C \exp(Ct). \quad (6)$$

L_k^p norm. The estimate for the L_k^p norm $\|U\|_{L_k^p}$ can be obtained from (4) and (6) by induction on real values of k in increments less than 1. The proof proceeds as in [3, §V, 12] and we have $\|U\|_{L_k^p(M \times [t, t+\delta])} \leq C \exp(Ct)$.

In consequence, all the L_k^p norms of U remain bounded on $M \times [0, \omega)$, since the constants C are independent of $t \in [0, \omega)$; C may depend on δ , but this does not concern us, for δ is fixed once and for all. Hence, by the Sobolev imbedding theorem [3, §II, 13], all the derivatives $(\partial/\partial t)' \nabla^s U$ of U are uniformly bounded on $M \times [0, \omega)$. This completes the proof.

4. Remarks. The reason for assuming the boundary $\partial N'$ of N' to be convex in Theorem 1 can be found in [3, §IV, 3]. Theorem 1 remains valid even when N' is without boundary, $\partial N' = \emptyset$.

There are several examples of harmonic maps $F: K \rightarrow K'$, $\tau(F) = 0$, with constant energy density $e(F)$. For instance: the identity maps when $K = K'$, minimal immersions of Riemannian manifolds and the Hopf fibrations $S^3 \rightarrow S^2$, $S^7 \rightarrow S^4$, $S^{15} \rightarrow S^8$ in their classical polynomial representations. See [2].

By slight modifications of the proof of Theorem 1, we can prove a similar result for the Neumann problem. In fact, we have

THEOREM 2. *Let $M = N \times_{\varphi} K$, $M' = N' \times_{\psi} K'$ and $F: K \rightarrow K'$ be as in Theorem 1. Let $f_0: N \rightarrow N'$ be a C^∞ map with $\partial f_0/\partial \nu = 0$ on ∂N . Then the Neumann initial-boundary value problem for the nonlinear parabolic equation of Eells-Sampson*

$$\begin{aligned} \partial U/\partial t &= \tau(U) && \text{on } M \times [0, \infty), \\ U &= f_0 \times F && \text{on } M \times 0, \\ \partial U/\partial \nu &= 0 && \text{on } \partial M \times [0, \infty) \end{aligned} \tag{N}$$

has a global solution $U: M \times [0, \infty) \rightarrow M'$, which is unique, and C^∞ except on the corner $\partial M \times 0$.

We refer to [3, §V, 14] for an indication of necessary modifications of the proof.

Compare Theorem 2 with Theorem 1 of [5], in which an estimate for the existence time of solutions of the Neumann problem (N) is given in terms of the L^∞ norm of initial values and the bounds of the sectional curvatures of the source and target manifolds.

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