ENTIRE FUNCTIONS THAT SHARE REAL ZEROS AND REAL ONES

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ABSTRACT. We find all pairs of nonconstant entire functions that have the same zeros and ones (ignoring multiplicities), when the zeros and ones are all real.

We say two entire functions $f(z)$ and $g(z)$ share the value c provided that $f(z) = c$ if and only if $g(z) = c$. We distinguish between sharing a value **CM** (counting multiplicities) and **IM** (ignoring multiplicities). Unless stated otherwise, all functions will be assumed to be nonconstant and entire.

It is easy to show that if f and g share 0 and 1 **CM** and $f \neq g$, then there are entire functions α and β such that

$$f(z) = \frac{e^{2\pi i \alpha(z)} - 1}{e^{2\pi i \beta(z)} - 1} \quad \text{and} \quad g(z) = \frac{e^{-2\pi i \alpha(z)} - 1}{e^{-2\pi i \beta(z)} - 1}. \quad (1)$$

When f and g have finite order, then C. F. Osgood and C. C. Yang [5, p. 410] have shown that β is a polynomial and α is a polynomial in β with rational coefficients. Thus all possible pairs are determined. For infinite order, the example $\alpha(z) = z \sin(\pi z^2)$ and $\beta(z) = z^2$ shows that α does not even have to be a power series in β .

When f and g share 0 and 1 **IM**, the situation is more complicated. One of the authors [3, Theorem 3] has shown that $T(r, f) < (3 + o(1))T(r, g)$ and $T(r, g) < (3 + o(1))T(r, f)$ as $r \rightarrow \infty$ outside a set of finite linear measure ($T(r, h)$ is the Nevanlinna characteristic function of h). M. Ozawa [6] has proven some uniqueness theorems when f and g have finite order, by assuming various further hypotheses on the zeros and ones. But in general, no one has come close to finding all possible pairs.

Let $\mathcal{H}$ be the class of all nonconstant entire functions which have only real zeros and real ones. We will prove the following:

THEOREM. *If f and g are in $\mathcal{H}$ and share 0 and 1 **IM**, then we necessarily have one of the following six cases where $a \neq 0$ and b are real constants:*

1. $f = g$,
2. $f(z) = \sin^2(az + b)$ and $g(z) = -i \sin(az + b)e^{i(az + b)}$,
3. $f(z) = \frac{\sin^2(p(az + b))}{\sin^2(az + b)} \quad \text{and} \quad g(z) = \frac{\sin(p(az + b))}{\sin(az + b)} e^{i(p-1)(az + b)}$
for $p = -2$ and for $p = -3$,

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$$4. \quad f(z) = 1 - \frac{(g(z) - 1)^2}{e^{2\pi i(az+b)}} \quad \text{and} \quad g(z) = \frac{\sin(p(az+b))}{\sin(az+b)} e^{i(p-1)(az+b)}$$

for $p = 3$ and for $p = 4$.

REMARKS. Each case is distinct and it can be readily verified that the five unequal pairs are in $\mathcal{H}$ and share 0 and 1 **IM**. The following three corollaries are immediate consequences of the theorem. We note that the first part of the proof of the theorem is a direct proof of Corollary 2.

COROLLARY 1. *If f and g are in $\mathcal{H}$ and share 0 and 1 **CM**, then $f = g$.*

COROLLARY 2. *If f and g are in $\mathcal{H}$ and share 0 and 1 **IM**, and they are also both real on the real axis, then $f = g$.*

COROLLARY 3. *If $f(z)$ and $\sin z$ share 0 and 1 **IM**, then $f(z) \equiv \sin z$.*

L. A. Rubel and C. C. Yang [7, p. 293] proved the statement of Corollary 3 with “**IM**” replaced by “**CM**”. See also Theorem 1 of [6].

We mention the systematic study of meromorphic functions that share values by R. Nevanlinna in [4].

To prove our theorem we will apply results of Laguerre and A. Edrei.

LEMMA 1 [8, p. 266]. *If $h(z)$ is entire, real for real z , of order less than two, and with only real zeros, then $h'(z)$ has exactly one real zero between consecutive zeros of $h(z)$, and this zero is simple.*

LEMMA 2 [2]. *If $h(z)$ is entire with only real (positive) zeros and only real (positive) ones, then the order of h is at most one (one-half). If further, $h(z)$ is not real for some real value of z , then h is necessarily of one of the following two forms:*

$$h(z) = \frac{\sin(az+b)e^{i(az+c)}}{\sin(b-c)}, \quad \sin(b-c) \neq 0, \quad (i)$$

where a , b , and c are real constants;

$$h(z) = \frac{\sin(p(az+b))e^{i(p-1)(az+b)}}{\sin(az+b)} \quad (ii)$$

where $p \neq 0$, 1 is an integer and a , b are real constants.

PROOF OF THE THEOREM. Let f and g be in $\mathcal{H}$ and share 0 and 1 **IM**. Then by Lemma 2, $\text{order}(f) < 1$ and $\text{order}(g) < 1$.

Assume first that both f and g are real for real z . Let x_1 and x_2 be consecutive zeros of f , g . We deduce from Lemma 1 that, in the open interval (x_1, x_2) , both f and g have exactly one of the following: (i) one double one-point, (ii) two simple one-points, or (iii) no one-points. By considering $1 - f$ and $1 - g$, we see that the same statement can be made with the zeros and ones interchanged. Therefore, between any two zeros (ones), all of the ones (zeros) are shared **CM**. We have two cases.

In the first case, suppose that $f(x) \neq 0$ for $x > x_3$. It can be seen from the proof of Lemma 1 that f'/f is strictly decreasing on the interval $x > x_3$ (Note: In the proof of Lemma 1 it is shown that h'/h is strictly decreasing where it is defined on the real axis, for any h satisfying the hypothesis of Lemma 1.). Hence, f' can have at most one zero in (x_3, ∞) . Therefore, f can have at most two one-points in (x_3, ∞) . It follows from Lemma 2 that $\text{order}(f) < \frac{1}{2}$ and $\text{order}(g) < \frac{1}{2}$. If we further have that $f(x) \neq 0$ for $x < x_4$, then f and g must be polynomials by the Hadamard factorization theorem. This implies that $f = g$, because two polynomials that share two finite values **IM** are identical [1]. On the other hand, if f does have infinitely many negative zeros, then by the previous paragraph, all but at most two of the one-points will be shared **CM**. Thus by the Hadamard factorization theorem, there exists a rational function $R(z)$ such that

$$f - 1 = R(g - 1).$$

Since $R(z) = 1$ whenever z is a zero of f, g , this implies that $R(z) \equiv 1$. Hence $f = g$.

If in the preceding paragraph we had started with the assumption that $f(x) \neq 0$ for $x < x_4$, instead of $f(x) \neq 0$ for $x > x_3$, then the same argument will yield $f = g$. Similarly, if the assumption was either $1 - f(x) \neq 0$ for $x > x_3$, or $1 - f(x) \neq 0$ for $x < x_4$, then the same argument with $1 - f$ and $1 - g$ instead of f and g will again yield $f = g$.

The second case is when f and g have infinitely many positive zeros, infinitely many negative zeros, infinitely many positive ones, and infinitely many negative ones. Our previous application of Lemma 1 to the zeros of f, g and also to the zeros of $1 - f, 1 - g$, shows that f and g share 0 and 1 **CM**. Hence if $f \neq g$, then the result of Osgood and Yang [5, p. 410] gives

$$f(z) = \frac{e^{n(az+b)} - 1}{e^{az+b} - 1} \quad \text{and} \quad g(z) = \frac{e^{-n(az+b)} - 1}{e^{-(az+b)} - 1}, \quad (2)$$

where $a \neq 0, b$ are constants and $n \neq 0, 1$ is an integer. Now suppose f and g have a zero $z_1 = (2\pi ik/na) - (b/a)$ (k is an integer $\neq 0, \text{mod}(n)$). Then $z_2 = z_1 + (2\pi i/a)$ is also a zero. Since z_1 and z_2 are real, we obtain $\text{Re}(a) = \text{Re}(b) = 0$. But then f and g will not be real for all real z . Hence there are no zeros. But the same argument with $1 - f$ and $1 - g$ will show there are no ones. By Picard's theorem, f and g are constants, a case we have excluded. Therefore the assumption is false and $f = g$.

Now suppose that neither f nor g is real for all real z . Then each must have one of the forms in Lemma 2. It follows that $f = g$.

The last possibility is when, say, f is real for real z and g is not real for some real value of z . Then g is one of the two forms in Lemma 2.

Case 1. $g(z) = \sin(az + b)e^{i(az+c)}/\sin(b-c)$ where $a \neq 0, b, c$ are real constants. Then g has all simple zeros which occur when $az + b = \pi m$ for an integer m and all simple ones which occur when $az + c = \pi N$ for an integer N . Since the distance between any two consecutive zeros and any two consecutive ones is equal, there is exactly one zero between every two ones. By applying Lemma 1 to $1 - f$,

we see that f has all double zeros. This means

$$f(z) = (g(z))^2 e^{Az+B} \quad \text{for constants } A, B.$$

Since f is real on the real axis, this means that there are real constants C, D , and an integer k so that

$$f(z) = \frac{\sin^2(az+b)}{\sin^2(b-c)} e^{Cz+D+\pi ik}.$$

Now $g(z) = 1 \Rightarrow \sin^2(az+b) = \sin^2(b-c) \Rightarrow e^{Cz+D+\pi ik} = 1$. Since the ones are all real, it follows that $e^{Cz+D+\pi ik} \equiv 1$. Further, since the ones of f can only occur when $az+c = \pi N$, this means that $\sin^2(b-c) = 1$. Thus

$$f(z) = \sin^2(az+b) \quad \text{and} \quad g(z) = -i \sin(az+b) e^{i(az+b)}.$$

Case 2. $g(z) = \sin(p(az+b)) e^{i(p-1)(az+b)} / \sin(az+b)$ where $p \neq 0, 1$ is an integer and $a \neq 0, b$ are real constants. Then g has all simple zeros which occur when $p(az+b) = \pi N$ for an integer $N \neq 0 \pmod{p}$, and all simple ones which occur when $(p-1)(az+b) = \pi m$ for an integer $m \neq 0 \pmod{p-1}$.

Suppose first $p < 0$. There is at most one zero between consecutive ones. Applying Lemma 1 to $1-f$ shows that f can have only double zeros. Then for constants c, d

$$f(z) = (g(z))^2 e^{cz+d}.$$

Since f is real on the real axis, this means there are real constants A, B and an integer k so that:

$$f(z) = \frac{\sin^2(p(az+b)) e^{Az+B+k\pi i}}{\sin^2(az+b)}.$$

Since the ones are real, this further reduces to

$$f(z) = \frac{\sin^2(p(az+b))}{\sin^2(az+b)}.$$

If $p = -1$ then $f = 1$, the excluded constant case. The cases $p = -2$ and $p = -3$ give item 3 in the theorem. Now assume $p \leq -4$. If $az+b = \pi/(p+1)$, then $f(z) = 1$. Hence $g(z) = 1$, and from the preceding paragraph, this means that $(p-1)/(p+1) = m$ for some integer m . This cannot happen for $p \leq -4$.

Now suppose $p \geq 2$. There is at most one one-point between zeros. Hence by Lemma 1, f can have only double ones. Then for constants c and d :

$$f(z) - 1 = -(g(z) - 1)^2 e^{cz+d}.$$

By considering two different zeros of f, g we obtain $\operatorname{Re}(c) = \operatorname{Re}(d) = 0$. Then for real constants A, B ,

$$f(z) = 1 - (g(z) - 1)^2 e^{i(Az+B)}. \quad (3)$$

From this we obtain

$$f(z) = 1 - (w(z))^2 \exp[i(Az+B+2p(az+b))]$$

where $w(z) = \sum_{k=1}^{p-1} \exp[i(p-2k)(az+b)]$ is real for real z . Hence

$$\exp(i(Az+B)) \equiv \exp(-2pi(az+b)).$$

Substituting this into (3),

$$f(z) = 1 - (g(z) - 1)^2 e^{-2pi(az+b)}.$$

Then for $p = 2$, $g(z) = e^{2i(az+b)} + 1$ and $f(z) \equiv 0$. The cases $p = 3$ and $p = 4$ give item 4 in the theorem. Now assume $p \geq 5$. $f(z) = 0 \Leftrightarrow (g(z) - 1)^2 = e^{2pi(az+b)}$. Thus $f(z) = g(z) = 0$ whenever $az + b = \pi/(p-2)$. Therefore, $p = N(p-2)$ for some integer N . This is impossible for $p \geq 5$.

The proof of the theorem is now complete.

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