DIMENSION OF SUBSETS OF PRODUCT SPACES

Y. STERNFELD

ABSTRACT. It is proved that under a certain condition on a separable metric space X , each compact subset W of $X \times Y$ with $\dim W = \dim X + \dim Y$ contains a product $X' \times Y' \subset W$ with $\dim X' = \dim X$ and $\dim Y' = \dim Y$. This condition is satisfied when X is a Euclidean space.

A classical result of dimension theory states that a subset of the Euclidean n space $\mathbf{R}^n$ has dimension n if and only if it has a nonempty interior in $\mathbf{R}^n$. A different way to state the same result is the following: For $W \subset \mathbf{R}^n$, $\dim W = n$ if and only if W contains an n -dimensional rectangle, i.e., a product $\prod_{i=1}^n I_i$ where I_i is a nondegenerate interval, $1 \leq i \leq n$. In this note we wish to extend this result (in its second statement) to product spaces other than $\mathbf{R}^n$, and study the following question:

Let X and Y be finite-dimensional separable metric spaces, and let W be a subset of $X \times Y$ with $\dim W = \dim X + \dim Y$. Under which conditions will it follow that W contains a "rectangle" of the form $X' \times Y'$ with $\dim X' = \dim X$ and $\dim Y' = \dim Y$.

We shall see that compactness of W and a certain countability condition on one of the factors X or Y (which is satisfied when X (or Y) is Euclidean) imply a positive answer to our question, while without compactness of W it fails even in the simplest possible case, namely when X is an interval and Y the Cantor set. We do not know what the answer to this problem is if we just assume compactness of W .

To state our main result we introduce the following terminology:

DEFINITION. Let X be a separable metric space.

1. A family $\mathcal{Q}$ of closed subsets of X is said to be a *dimensional network* for X if $\dim H = \dim X$ for all $H \in \mathcal{Q}$, and if each closed $F \subset X$ with $\dim F = \dim X$ contains some element of $\mathcal{Q}$.

2. X is said to be *countably n -dimensional* if $\dim X = n$ and it has a countable dimensional network.

Note that a space is countably zero dimensional if and only if it is countable, that the space $\mathbf{R}^n$ is countably n -dimensional, and that for any n -dimensional space X , $X \times \Delta$ is an n -dimensional space which fails to be countably n -dimensional, where Δ denotes the Cantor set.

It turns out that the above question has a positive answer if W is compact and one of the factors X or Y has its dimension countably.

Received by the editors February 20, 1980 and, in revised form, October 29, 1980.
 AMS (MOS) subject classifications (1970). Primary 54F45.

© 1981 American Mathematical Society
 0002-9939/81/0000-0328/\$01.75

THEOREM 1. *Let X and Y be separable metric spaces with X countably ($\dim X$)-dimensional, and let $W \subset X \times Y$ be compact with $\dim W = \dim X + \dim Y$. Then there exist compact sets $X' \subset X$, $Y' \subset Y$, with $\dim X' = \dim X$, $\dim Y' = \dim Y$ and $X' \times Y' \subset W$.*

The compactness of W in Theorem 1 cannot be omitted. It follows from the theorem that it will still hold if we assume that W is σ -compact. But it turns out that Theorem 1 fails if we assume that W is an absolute G_δ . Indeed, in [3, pp. 80–81], following Knaster, Lelek constructs an n -dimensional totally disconnected G_δ subset of $\mathbb{R}^{n+1}$. It easily follows from the construction that this set is homeomorphic to a subset W of $I^n \times \Delta$, such that for each $t \in \Delta$, $(I^n \times \{t\}) \cap W$ consists of a single point. In particular W does not contain any set of the form $X \times Y$, $X \subset I^n$, $Y \subset \Delta$, for which X contains more than one point. (Here $I = [-1, 1]$ and Δ is the Cantor set.)

As mentioned above we do not know whether the countability condition in Theorem 1 is really needed; we state this as a problem:

PROBLEM 1. Let W be a compact subset of $X \times Y$ with $\dim W = \dim X + \dim Y$ (X, Y separable metric spaces). Does it follow that W contains a product $X' \times Y'$ with $\dim X' = \dim X$ and $\dim Y' = \dim Y$? In particular does the above hold if $\dim X = \dim Y = 1$? (If $\dim X$ or $\dim Y = 0$ the answer is positive.)

For the proof of Theorem 1 we shall need the following result of Jung and Keesling [1], [2].

THE JUNG-KEESLING THEOREM. *Let P be a closed mapping of an $(n + k)$ -dimensional metric space W onto a k -dimensional space Y . Set $D = \{y \in Y: \dim P^{-1}(y) > n\}$. Then $\dim W = n + k < \dim D + \dim P$, where $\dim P = \sup_{y \in Y} \dim P^{-1}(y)$.*

PROOF OF THEOREM 1. Set $\dim X = n$, $\dim Y = k$. Let P denote the restriction to W of the natural projection of $X \times Y$ onto Y . For $y \in Y$, $P^{-1}(y)$ is homeomorphic to a subset of X , hence $\dim P^{-1}(y) \leq n$, i.e., $\dim P \leq n$. The compactness of W implies that P is a closed mapping, and by the Jung-Keesling Theorem, $\dim D > (n + k) - n = k$, where D is the set defined in this theorem. For a closed subset F of X set $\varphi(F) = \{y \in Y: F \times \{y\} \subset W\}$. $\varphi(F)$ is closed in Y , since by its definition $F \times \varphi(F) \subset W$, and by the compactness of W , $F \times \overline{\varphi(F)} \subset \overline{F \times \varphi(F)} \subset W$.

Let $\mathcal{Q}$ be a countable dimensional network for X . We claim that $D \subset \bigcup_{F \in \mathcal{Q}} \varphi(F)$. Indeed, let $y \in D$. Then $\dim P^{-1}(y) = n$, and by the definition of a dimensional network there exists some $F \in \mathcal{Q}$ such that $P^{-1}(y) = (X \times \{y\}) \cap W \supset F \times \{y\}$. In particular, $F \times \{y\} \subset W$ and $y \in \varphi(F)$.

It follows that $\dim \bigcup_{F \in \mathcal{Q}} \varphi(F) \geq \dim D > k$, and since the sets $\varphi(F)$ are closed, $\dim \varphi(F) > k$ for some $F \in \mathcal{Q}$. We also have $\dim F = n$ and $F \times \varphi(F) \subset W$ and the theorem follows. $\square$

Finally, let us state two more problems both related to a possible inequality in the product theorem.

PROBLEM 2. Can one have $\dim(X' \times Y') = \dim W$ in Theorem 1 and in Problem 1?

PROBLEM 3. What can be said if the condition $\dim W = \dim X + \dim Y$ is replaced by $\dim W = \dim(X \times Y)$ in Theorem 1 and Problem 1?

REFERENCES

1. C. F. K. Jung, *Mappings on compact metric spaces*, Colloq. Math. **19** (1968), 73–76.
2. J. Keesling, *Closed mappings which lower dimension*, Colloq. Math. **20** (1969), 237–241.
3. A. Lelek, *Dimension inequalities for unions and mappings of separable metric spaces*, Colloq. Math. **23** (1971), 69–91.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAIFA, HAIFA, ISRAEL