NONTORAL PINCHED ANOSOV MAPS

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ABSTRACT. We show Anosov maps satisfying a certain spectral property can arise on nilmanifolds of any nilpotent degree, settling a conjecture of Brin.

Brin studied Anosov maps on closed Riemannian manifolds satisfying a certain condition on the spectrum of the operator induced on vector fields [3]. He showed that such an $A: M \rightarrow M$ is transitive and that M is covered by Euclidean space and he conjectured that M must be a torus. We show that, on the contrary, Anosov automorphisms satisfying Brin's condition may be constructed on nilmanifolds of any nilpotent degree.

This result for degree 2 is joint work with S. Dani and we were helped greatly by conversations with R. Szczarba. We also thank Yale University for supporting this research.

Dani has pointed out that the examples constructed below for degree > 2 are new examples of Anosov diffeomorphisms.

Brin's condition is that the spectrum of the operator $A_*: X(M) \rightarrow X(M)$ defined by $(A_*(v))(m) = DA(A^{-1}m)(v(m))$ lies in the union of two annuli $\{x \in \mathbb{C} | 0 < r_1 < |x| < r_2 < 1 \text{ or } 1 < R_2 < |x| < R_1 < \infty\}$ satisfying

$$1 + \frac{\log R_2}{\log R_1} > \frac{\log r_1}{\log r_2}. \quad (*)$$

By work of Mather [6], A is Anosov iff the spectrum of A_* lies off the unit circle; if $(*)$ holds, we will call A *pinched*.

The construction is based on the scheme of Auslander and Scheuneman [1]. Given d we will construct a semisimple matrix $A_1 \in \text{SL}(d, \mathbb{Z})$ with special eigenvalues. This A_1 extends to a Lie automorphism A_k of the free k -step nilpotent Lie algebra with d generators $N_k(\mathbb{R}^d)$. For a certain ideal I , A_k induces a lattice-preserving automorphism $\bar{A}$ of $N_k(\mathbb{R}^d)/I$. If $k < \sqrt{d}$, this map $\bar{A}$ induces a pinched Anosov automorphism A on a nilmanifold of nilpotent degree exactly k .

1. The construction of A_1 . Let $d > 2$ be an integer. Let F be a totally real algebraic number field of degree d over $\mathbb{Q}$. By the geometric version of the Dirichlet Unit Theorem [2], the map $l: F^* \rightarrow \{(y_i) \in \mathbb{R}^d | \sum y_i = 0\} = V$, $l(x) = (\log|x^{(i)}|)$ where $x = x^{(1)}, x^{(2)}, \dots, x^{(d)}$ are the conjugates of x , carries the algebraic units of F onto a lattice $L_0 \subset V$. By going far out along the ray through

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$(1 - d, 1, \dots, 1) \in V$ and choosing a nearby lattice point $l(x)$, one obtains a unit x satisfying $|\log|x^{(i)}|/\log \lambda - 1| < \varepsilon$ for some $\lambda > 1$, $i = 2, \dots, d$, where $\varepsilon = \varepsilon(d)$ will be chosen later.

The unit x acts on $\mathbf{R}^d$ and preserves the lattice L_0 via $(z^{(i)}) \mapsto (x^{(i)}z^{(i)})$. Let $A_1 \in \text{SL}(d, \mathbf{Z})$ be the matrix representing this action relative to an integral basis for L_0 . Clearly the original coordinate directions are eigenvectors for A_1 with eigenvalues $x^{(1)}, \dots, x^{(d)}$, so A_1 is semisimple. Moreover, the property that $|x^{(i)}| \approx \lambda > 1$ for $i > 1$ implies that the characteristic equation of A_1 , $\prod_i (t - x^{(i)})$, is irreducible. Hence the $x^{(i)}$ are distinct. The eigenvector e_1 corresponding to $x^{(1)}$ may be chosen to have coordinates in $\mathbf{Q}(x^{(1)})$. Conjugation of these coordinates gives eigenvectors $e_2, \dots, e_d$ corresponding to the eigenvalues $x^{(2)}, \dots, x^{(d)}$.

2. Construction of the Lie automorphism $\bar{A}$. For $k \geq 2$, let $N_k(\mathbf{R}^d)$ be the free k -step nilpotent Lie algebra on d generators. Let $I = I(A_1) \subset N_k(\mathbf{R}^d)$ be the vectorspace spanned by all products of eigenvectors e_i in which some e_i appears at least twice. Clearly I is an ideal and is invariant under the Lie automorphism A_k of $N_k(\mathbf{R}^d)$ obtained by extending the action of A_1 on $\mathbf{R}^d$.

We will show that, relative to the natural integral lattice $L_k \subset N_k(\mathbf{R}^d)$, I is spanned by integral vectors. Clearly I is spanned by those products consisting of 2 copies of e_i , some $i = 1, \dots, d$, and the other factors integral vectors. Fixing a choice of integral vectors and an order in which the multiplication is to be performed and letting e_i vary over the eigenbasis chosen in §1 gives vectors $s_1, \dots, s_d \in I$. Note that s_1 is defined over $\mathbf{Q}(x^{(1)})$, so $s_1 = \sum_{p=0}^{d-1} (x^{(1)})^p v_p$ where v_p is a rational vector. Since the e_i , $i > 1$, are conjugates of e_1 , we have $s_i = \sum_{p=0}^{d-1} (x^{(i)})^p v_p$. The transformation $((x^{(i)})^p)$ carrying $(v_p)_{p=0}^{d-1}$ to $(s_i)_{i=1}^d$ is invertible, since it is a Vandermonde matrix and the $x^{(i)}$ are distinct. Thus the span of the s_i equals the span of the v_p and I is spanned by rational vectors v_p .

Thus L_k meets I in a sublattice of rank equal to $\dim I$, and so the integral vectors in $\bar{N} = N_k(\mathbf{R}^d)/I$ form a lattice Γ . It is clear that A_k induces an automorphism $\bar{A}$ of $\bar{N}$ which preserves Γ .

3. Hyperbolicity and pinching. We show that for the proper choice of $\varepsilon(d)$ and for $k < \sqrt{d}$ that $\bar{A}$ is hyperbolic and its eigenvalues satisfy the pinching condition. Once these facts are shown, it will follow that $\bar{A}$ induces a hyperbolic automorphism $\tilde{A}$ of the simply connected Lie group G determined by $\bar{N}$. The theory of Malcev [5] implies that $\exp(\Gamma')$ is a lattice, for some Γ' of finite index in Γ . Since the lattices mL_1 , $m \geq 1$, generate a cofinal collection Γ'_m of sublattices of Γ , we may assume $\Gamma' = \Gamma'_m$. But since Γ'_m is isomorphic to $\Gamma = \Gamma'_1$ under the automorphism of $\bar{N}$ induced by multiplication by m on $\mathbf{R}^d$, it must be that $\exp(\Gamma)$ is itself a lattice. Hence $\bar{A}$ induces an automorphism A of $M = G/\exp(\Gamma)$. Using the parallelization on M by left-invariant vector fields, it is easy to check that the pinching condition on the eigenvalues of $\bar{A}$ implies that A is pinched.

The eigenvectors (resp. eigenvalues μ) of $\bar{A}$ are all products of $\leq k$ distinct eigenvectors e_i (resp. eigenvalues $x^{(i)}$), by our choice of the ideal I . Thus one may

estimate, according to whether e_1 occurs or not,

$$\log r_1 = \log |x^{(1)}| \leq \log |\mu| \leq \log |x^{(1)}| + (k-1) \sup \log |x^{(i)}| = \log r_2$$

or

$$\log R_2 = \inf_{i \geq 1} \log |x^{(i)}| \leq \log |\mu| \leq k \cdot \sup \log |x^{(i)}| = \log R_1.$$

Taking logs to the base λ and using $d > k^2$, one has

$$\begin{cases} \log r_2 \approx -(d-k) < 0 \\ \log R_2 \approx 1 > 0, \end{cases}$$

and

$$\left\{ \frac{\log r_1}{\log r_2} \approx \frac{-(d-1)}{-(d-k)} < 1 + \frac{1}{k} \approx 1 + \frac{\log R_2}{\log R_1} \right\}.$$

So for $\epsilon(d)$ sufficiently small, $\bar{A}$ is hyperbolic and the desired pinching condition holds.

Finally we must check that the k th term $C^k(\bar{N})$ in the central series of $\bar{N}$ is nontrivial. However, it is clearly spanned by all products of k distinct eigenvectors from the eigenbasis $e_1, \dots, e_d$. By [4, 11.2.4] one may pick a basis by choosing any k distinct e_i in any cyclic order and introducing brackets in a standard way. It follows that $C^k(\bar{N})$ has dimension $\binom{d}{k} \cdot (k-1)!$, which is positive for $d > k$.

Thus for $d > k^2$ all the desired properties hold.

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